ARITHMETIC

AND

ALGEBRA,

DESIGNED FOR THE USE OF THOSE WHO ARE NOT CANDIDATES
FOR HONORS,

WITH A VARIETY OF EXAMPLES,

AND ALL THE

SENATE-HOUSE EXAMINATION PAPERS

THAT HAVE BEEN PROPOSED
SINCE THE COMMENCEMENT OF THE NEW SYSTEM IN 1841.

BY

HENRY PIX, B.A.

EMMANUEL COLLEGE, CAMBRIDGE.

CAMBRIDGE:

PRINTED BY AND FOR J. HALL, OPPOSITE THE PITT PRESS;

AND G. BELL, 186, FLEET STREET, LONDON.

M.DCCC.XLIV.

ADVERTISEMENT.

THE present Publication has been drawn up to meet the wishes of the Candidates for the ordinary B.A. Degree, and comprises as much as is necessary for them to read in the subjects of Arithmetic and Algebra. The principal things attempted in it are, to give all the Rules that are required, with a collection of Examples under each; and to explain the reason of all the operations, as they occur. Attention has been constantly paid to the Papers, which have been given of late years, and it is therefore hoped that no Example may be found in them, which cannot be readily worked out by referring to that part of the book, which treats of that class in particular.

October, 1844.

CONTENTS.

ARITHMETIC.

ALGEBRA.

ERRATA.

Page	Line	Error.	Correction.
61,	12,	fraction $\frac{12}{7}$	fraction is $\frac{12}{7}$
93,	Ex. 8,	$1621\frac{1}{4}$	560.224
98,	Ex. 17,	£99. 15*s*. $7\frac{1}{2}$*d*.	£99. 5*s*. $11\frac{1}{4}$*d*.
177,	Ex. 33,	$\sqrt{4x=\sqrt{7x+2}}$	$\sqrt{4x-\sqrt{7x+2}}$

SCHEDULE OF ARITHMETIC AND ALGEBRA OF EXAMINATION,

FOR THE DEGREE OF A. B.

OF PERSONS NOT CANDIDATES FOR HONORS.

ARITHMETIC.

Addition, subtraction, multiplication, division, reduction, rule of three; the same rules in vulgar and decimal fractions: practice, simple and compound interest, discount, extraction of square and cube roots, duodecimals.

ALGEBRA.

1. Definitions and explanation of algebraical signs and terms.

2. Addition, subtraction, multiplication and division of simple algebraical quantities and simple algebraical fractions.

3. Algebraical definitions of ratio and proportion.

4. If $a : b :: c : d$ then $ad = bc$, and the converse:

$$\begin{aligned} \text{also } b &: a :: d : c, \\ \text{and } a &: c :: b : d, \\ \text{and } a + b &: b :: c + d : d. \end{aligned}$$

5. $$\begin{aligned} \text{If } a &: b :: c : d, \\ \text{and } c &: d :: e : f, \\ \text{then } a &: b :: e : f. \end{aligned}$$

6. $$\begin{aligned} \text{If } a &: b :: c : d, \\ \text{and } b &: e :: d : f, \\ \text{then } a &: e :: c : f. \end{aligned}$$

7. Geometrical definition of proportion. (EUC. Book v. Def. 5.)

8. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

9. Definition of a quantity *varying as* another, *directly*, or *indirectly*, or as two others *jointly*.

A

ARITHMETICAL TABLES.

1.—TABLE OF MONEY.

4 farthings make 1 penny (*d.*)
12 pence .... 1 shilling (*s.*)
20 shillings .... 1 pound (£.)

Groat is equal to 4*d.*
Tester .... 6*d.*
Crown .... 5*s.*

Gold Coins.

Noble is equal to 6*s.* 8*d.*
Angel .... 10*s.*
Mark .... 13*s.* 4*d.*
Pistole .... 17*s.*
Sovereign .... 20*s.*
Guinea .... 21*s.*
Carolus .... 23*s.*
Jacobus .... 25*s.*
Moidore .... 27*s.*

2.—WEIGHT.

Troy Weight.

24 grains (*grs.*) make 1 pennywt. (*dwt.*)
20 dwts .... 1 ounce (*oz.*)
12 ozs .... 1 pound (*lb.*)

Avoirdupois Weight.

16 drams (*dr.*) make 1 oz.
16 ozs. .... 1 lb.
14 lbs. .... 1 stone, (*st.*)
28 lbs. .... 1 quarter, (*qr.*)
4 qrs. .... 1 hundwt. (*cwt.*)
20 cwt. .... 1 ton.

Apothecaries' Weight.

20 grs. make 1 scruple, (*scr.*)
3 scrs. .... 1 dr.
8 drs. .... 1 oz.
12 ozs. .... 1 lb.

N.B.—A firkin (of butter) equals 56 lb.
A pack (of wool) .... 240 lb.

3.—LINEAR MEASURE.

12 ins. make 1 foot, (*ft.*)
3 ft. .... 1 yard, (*yd.*)
5½ yds. .... 1 pole, (*pl.*) or rod or perch
40 pls. .... 1 furlong, (*fur.*)
8 furs. .... 1 mile, (*ml.*)
3 mls. .... 1 league.

3 barleycorns are equal to 1 inch.
A palm is .... 3 ins.
A hand .... 4 ins.
A span .... 9 ins.
A cubit .... 18 ins.
A pace .... 5 ft.
A fathom .... 6 ft.
A degree .... 69½ statute miles, or 60 geographical miles.

Cloth Measure.

2¼ ins. make 1 nail (*nl.*)
4 nls. .... 1 quarter, (*qr.*)
3 qrs. .... 1 Flemish ell.
4 qrs. .... 1 yard, (*yd.*)
5 qrs. .... 1 English ell.
6 qrs. .... 1 French ell.

4.—SUPERFICIAL MEASURE.

144 square ins. make 1 sq. ft.
9 sq. ft. .... 1 sq. yd.
30¼ sq. yds. .... 1 sq. pl. or perch, or rod.
40 sq. pls. .... 1 rood.
4 roods .... 1 acre.
640 acres .... 1 sq. mile.

5.—SOLID MEASURE.

1728 cubic in. make 1 cubic ft.
27 cubic ft. .... 1 cubic yd.

6.—MEASURE OF CAPACITY.

4 gills make 1 pint, (*pt.*)
2 pts. .... 1 quart (*qt.*)
4 qts. .... 1 gallon, (*gal.*)

Wine Measure.

42 gals. make 1 tierce.
63 gals. .... 1 hogshead, (*hhd.*)
84 gals. .... 1 puncheon.
2 hhds. .... 1 pipe.
2 pipes .... 1 tun.

Ale and Beer Measure.

9 gallons make 1 firkin, (*fir.*)
2 firs. .... 1 kilderkin, (*kil.*)
2 kil. or 36 gals. .... 1 barrel, (*bar.*)
1½ bar. or 54 gals. ... 1 hhd.
2 bar. .... 1 puncheon.
3 bar. or 2 hhd. .... 1 butt.

Corn or Dry Measure.

2 quarts	make	1 pottle.
2 pottles	....	1 gallon.
2 gals.	....	1 peck.
4 pecks	....	1 bushel.
8 bushels	....	1 quarter.
5 qrs.	....	1 load.

Coal Measure.

3 bushels	make	1 sack.
36 bushels	....	1 chaldron.

7.—TIME.

60 seconds	make	1 minute, (*min.*)
60 min.	make	1 hour, (*hr.*)
24 hrs.	....	1 day.
7 days	....	1 week, (*wk.*)
4 wks.	....	1 lunar month.
365 days	....	1 year.
12 calendar months	....	1 year.

8.—ANGULAR MEASURE.

60 seconds	make	1 minute.
60 min.	....	1 degree, (*deg.*)
90 deg.	....	1 quadrant.
4 quadrants	....	1 circumference, (O^{ce}.)

GENERAL REMARKS ON THE TABLES.

One farthing, 2 farthings, and 3 farthings are also denoted by the fractions $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, meaning one fourth, one half, and three fourths respectively of a penny.

The standard *gold* coin in this realm contains 22 parts of fine pure gold, 1 of fine silver, and 1 of fine copper. Each grain of gold is 2*d.* and each dwt. 4*s.*

The standard *silver* coin contains 37 parts of pure silver, and 3 of copper.

From a lb. avoirdupois of copper are coined 24 pence.

Troy weight is used for gold, silver, and jewellery, and in philosophical experiments; Avoirdupois weight for all common goods of a coarse or heavy nature; Apothecaries weight for mixing and preparing medical prescriptions; but drugs are *bought* by avoirdupois weight.

The Troy Pound is the standard at the Mint; and for Jewellers and Apothecaries. It is determined by the weight of a cubic inch of distilled water at a given temperature.

The division of the ounce into drams, the dram into scruples, and the scruple into grains, is not acknowledged in the Act.

The pound and ounce Apothecaries' weight and the pound and ounce Troy are the same, but the smaller divisions are different.

The yard regulates all measures, whether linear, superficial, or solid. It is determined by the length of a pendulum vibrating seconds, *in vacuo*, in the latitude of London, at the level of the sea.

Linear measure respects *length* only, and is used in taking the distance from one place to another.

It is probable that a barleycorn was formerly the element of *length*, but a barleycorn now is no measure.

Superficial measure respects both *length* and *breadth*, and is used in estimating the contents of all kinds of superficies or surfaces.

A *square* inch or *square* foot is the superficial content of a square, each of whose sides is a *linear* inch, and *linear* foot respectively.

Land is measured by a *chain* called Gunter's chain. It is 22 yards or 66 feet in length, and is divided into 100 equal parts or links. Ten chains in length and 10 in breadth, *i. e.* 10 square chains, make an acre.

Solid or cubic measure respects *length*, *breadth*, and *thickness* or *depth*. It is used in estimating the contents of solids, as in measuring stones, timber, &c. &c.

A *cubic* inch, or *cubic* foot, is the solid content of a cube, each of whose sides or edges is a *linear* inch and *linear* foot respectively.

The Imperial Gallon is now used both for liquids and dry goods. In contains about 277.2738 cubic inches or 277¼ inches nearly. Ten lb. avoirdupois of distilled water at a given temperature will exactly fill it.

Since 13 lunar months contain 364 days, these are often reckoned as a year.

All the 12 months, called *Calendar* months contain 31 days, except February which has 28, and April, June, September and November, which have 30 days. Every fourth year contains 366 days, and is called Leap Year, and in such a year February has 29 days. The common rule for finding whether a particular year will be leap year is to divide it by 4, and if nothing remain it is leap year. Thus 1848, 1852 will be leap years. If however the year be exactly a number of centuries as 1900, 2000, &c., then the year is a leap year only when the *century* is divisible by 4. Thus 2000, 2400, will be leap years, but 1900, 2100, will be common years. Julius Cæsar first introduced leap years, in order to adjust the calendar; and Pope Gregory, in the 16th century, introduced the modification of them, respecting the centuries, because under the former system, there was an error in excess of 3 days in 400 years.

The Gregorian correction is rather too great, but the error does not amount to a day in 4000 years.

ARITHMETIC.

CHAPTER I.

ELEMENTARY RULES.

ART. 1. ARITHMETIC is the science which treats of the properties of *numbers*, and the method of estimating the relations of quantities by means of them.

The number *one*, which is the standard fixed upon as the foundation of our various calculations, is called an *unit*: and the aggregate of any number of *ones* or *units*, is called an *integer*, or *whole number*.

2. The following *signs* or *abbreviations* are frequently made use of in Arithmetic. Thus

$\therefore$ is an abbreviation signifying *therefore*.

$=$ (equal) signifies that the two quantities between which it is placed are *equal* to each other. Thus 4 qrs. = 1 cwt.

$+$ (plus) is the sign of *addition*, and when placed between two quantities signifies that the latter is to be *added* to the former. Thus $6 + 4 = 10$.

$-$ (minus) is the sign of *subtraction*, and when placed between two quantities, signifies that the latter is to be *subtracted* from the former. Thus $6 - 4 = 2$.

$\times$ (into) is the sign of *multiplication*, and when placed between two quantities, signifies that they are to be *multiplied* together. Thus $4 \times 2 = 8$.

$\div$ (by) is the sign of *division,* and when placed between two quantities, signifies that the former is to be *divided* by the latter. Thus $4 \div 2 = 2$.

Division is also denoted by placing the quantities like a fraction; the upper number being the Dividend, and the lower the Divisor. Thus $\frac{112}{14} = 8$.

The common system of Notation and Numeration, and the fundamental operations of Addition, Subtraction, Multiplication, and Division of *Integers* and *Integral* Compound numbers are here supposed to be understood. The following examples are given for the sake of practice.

ADDITION.

ART. 3. Ex. 1. Find the sum of the numbers, 43, 401, 9747, 3464, 2263, 314, and 974. Ans. 17206.

2. Find the number of days in the 12 calendar months. Ans. 365.

3. Find the sum of the numbers 657890, 278679, 5798, 67843, 489567, 37429. Ans. 1537206.

4. Find the number of days in 4 successive years. Ans. 1461.

5. Find the sum of £2. 13*s.* $5\frac{1}{2}$*d.*; £7. 9*s.* $4\frac{1}{4}$*d.*; £5. 15*s.* $4\frac{1}{2}$*d.*; £9. 17*s.* $6\frac{1}{2}$*d.*; £7. 16*s.* 3*d.*; and £5. 14*s.* $7\frac{3}{4}$*d.* Ans. £39. 6*s.* $7\frac{1}{2}$*d.*

6. A service of plate consists of articles weighing respectively 203 oz. 8 dwts. 2 gr.; 408 oz. 9 dwts.; 112 oz. 8 dwts. 3 gr.; 71 oz. 7 dwts.; 73 oz. 5 dwts. 5 gr.; 121 oz. 4 dwts. 6 gr.; 131 oz. 7 dwts.; 105 oz. 5 dwts. 3 gr. What is the weight of the whole? Ans. 102 lb. 2 oz. 13 dwts. 19 grs.

7. Find the sum of 2 cwt. 3 qrs. 13 lb.; 2 cwt. 2 qrs. 12 lb. 4 oz.; 1 cwt. 3 qrs. 5 lb.; 3 cwt. 1 qr. 12 lb. 3 oz.; 2 cwt. 3 qrs. 15 lb. 5 oz. Ans. 13 cwt. 2 qrs. 1 lb. 12 oz.

9. Find the sum of [illegible] and [illegible]

Ans. [illegible]

10. Find the sum of [illegible] fur. [illegible] poles [illegible] yds. [illegible] ft.; [illegible] fur. [illegible] poles [illegible] yds. [illegible]; and [illegible] fur. [illegible] poles [illegible] yds. [illegible] in.

Ans. [illegible] miles [illegible] poles [illegible] yds. [illegible] ft. 1 in.

11. Find the sum of [illegible] mths. [illegible] wks. [illegible] days [illegible] hrs.; [illegible] mths. [illegible] wks. [illegible] days [illegible] hrs.; [illegible] mths. [illegible] wks. [illegible] days [illegible] hrs.; and [illegible] mths. [illegible] wks. [illegible] days [illegible] hrs.

Ans. [illegible] mths. [illegible] wk. [illegible] day 14 hrs.

---

4. In explaining the operations of Addition and the other simple Rules, it is necessary to bear in mind that every figure, besides its own proper value, expressing so many units, has also a *local* value, depending on its position. Thus the figure 5, *standing by itself*, or to the *right* hand of other figures, expresses 5 units. But if placed before any other figure, it expresses 5 *tens* of units.

Now in Addition, if the sum of the units exceed ten, or contain ten several times, we add the number of tens it contains to the next column, and only set down the number of units that are over. And the reason of this is obvious from the above consideration of the *local* value of figures, since an *unit* in any higher place, signifies the same thing as *ten* in the place immediately lower.

We may also remark that the common method of Addition of Integers is an abbreviated one. Thus, if we have to add together the numbers 354 and 467, the operation will be as follows:

$$
\begin{aligned}
354 &= 300 + 50 + 4 \\
467 &= 400 + 60 + 7 \\
\text{and the sum} &= 700 + 110 + 11 \\
&= 800 + 10 + 11 \\
&= 800 + 20 + 1 = 821
\end{aligned}
$$

by the principles of Notation.

The Addition of Integers, viewed in this way, is exactly analogous to the Addition of Money, Weight, &c. In the *latter*, the pounds, shillings, and pence are added by *themselves;* in the *former*, the hundreds, tens and units are added by *themselves.* In the *latter*, if the sum of a column be equal to, or exceed the number contained in an unit of the next superior order, we put down the excess, if any, and add the number of the next higher denomination contained in that sum to the next column; in the *former* if the sum of the units be equal to, or exceed 10, we put down the excess, if any, and add the number of tens it contains to the column of tens.

SUBTRACTION.

ART. 5. Ex. 1. The number of Chaldee verses in the Bible is 268, of which 200 occur in Daniel and 1 in Jeremiah—the rest are in Ezra; how many are they? Ans. 67.

2. How many years were there between the Creation and the Deluge (2348 B.C.) Ans. 1656.

3. From 90 thousand and 90 subtract 12345. Ans. 77745.

4. Subtract 475429 from 702306. Ans. 226877.

5. Subtract £218. 2*s.* $1\frac{3}{4}$*d.* from £316. 3*s.* $5\frac{1}{2}$*d.*
Ans. £98. 1*s.* $3\frac{3}{4}$*d.*

6. Subtract 29 gals. 3 qts. 1 pint from 35 gals. 1 qt. 0 pt.
Ans. 5 gals. 1 qt. 1 pt.

7. Subtract 19 yrs. 231 days 21 hrs. from 26 yrs. 213 days 11 hrs. Ans. 6 yrs. 346 days 14 hrs.

8. Subtract 39 qrs. 7 bus. 2 pks. from 45 qrs. 3 bus. 1 pk.
Ans. 5 qrs. 3 bus. 3 pks.

9. Subtract 13 yds. 1 qr. 3 nls. from 17 yds. 3 qrs. 2 nls.
Ans. 4 yds. 1 qr. 3 nls.

10. Subtract 23 acres 3 roods 35 poles from 37 acres 2 roods 29 poles. Ans. 13 acres 2 roods 34 poles.

ART. 6. The remarks made in Art. 4. enable us also to explain the operation of Subtraction, which is only the reverse of Addition. And the reason of *borrowing* 10 is obvious, if we consider that when two numbers are equally increased by adding the same to both, their difference will not be altered. Thus if we have to subtract 236 from 354 we add 10 to the 4 of the *minuend*, and 1 to the 3 of the *subtrahend*, which being in the higher place equals 10 of the lower place. In other words we take or *borrow* 1 ten out of the 5 tens and add it to the 4 units in the *minuend*, and then instead of supposing the 5 in the *minuend* diminished, we suppose the corresponding figure 3 in the *subtrahend* increased, which amounts to the same thing. Also in the subtraction of Compound Numbers, as Money, Weight, &c., &c., if the number to be subtracted exceed the number from which it is to be subtracted, we add to the latter as many as make one of the next higher denomination, instead of borrowing 10 as in abstract numbers, and then carry one to the next number of a higher denomination to be subtracted. The above example may be written thus at length:

$$354 = 300 + 50 + 4$$
$$= 300 + 40 + 14$$
$$236 = 200 + 30 + 6$$
$$\therefore 354 - 236 = 100 + 10 + 8 = 118 \text{ by the principles of Notation.}$$

MULTIPLICATION.

ART. 7. Ex. 1. Multiply 13795 by 34. Ans. 469030.

2. Multiply 771039 by 35. Ans. 26986365.

3. Multiply 391525 by 861. Ans. 337103025.

4. Multiply 3241 by 2700. Ans. 8750700.

5. Multiply £16. 6*s*. 10½*d*. by 48. Ans. £784. 10*s*. 0*d*.

6. Multiply 12 cwt. 2 qrs. 8 lb. by 5.
Ans. 62 cwt. 3 qrs. 12 lb.

7. Multiply 13 acres 3 roods 18 poles by 6.

Ans. 83 acres 28 poles.

8. Multiply 7 lb. 5 oz. 9 dwts. by 12.

Ans. 89 lb. 5 oz. 8 dwts.

9. Multiply 17 cub. yds. 21 ft. 57 in. by 84.

Ans. 1493 cub. yds. 11. ft. 1332 in.

10. Multiply £34. 8*s*. 2½*d*. by 3465.

Ans. £119232. 1*s*. 10½*d*.

Art. 8. Multiplication is the addition of the same number several times, and is a concise method of performing the operation of many additions.

The numbers multiplied in any case are called *factors*, and the result the *product*. In multiplying together the factors it matters not which we take as the *multiplier*.

The observations made concerning the *local* value of digits enable us to explain the operation of multiplication. When we multiply as in Ex. 1. by any figure (as 5 by 4) we carry the tens that are contained in the product to the *tens*, and similarly the hundreds to the hundreds. If the multiplier be any number with one or more cyphers to the right hand as in Ex. 4. we multiply by the number and annex an equal number of cyphers to the product. The reason of this is obvious from the use of cyphers in notation. The reason why we place the lines in a diagonal manner and not each under one another is also evident from the *local* value of the figures. Thus in Ex. 3. the reason why we place the first figure arising from the multiplication by 6 under the 6, and not under the 1, is because we are strictly speaking multiplying by 60, and therefore we must put the first figure under the column of *tens*. The cyphers ought properly to be placed in all these lines, but are omitted because they would make no difference when the lines are added up. Thus if we have to multiply 7329 by 365 the result would be as follows:

$$7329 \times 365 = 7329 \times (300 + 60 + 5)$$

$$\begin{aligned} \text{Now } 7329 \times 5 &= 36645 \\ 7329 \times 60 &= 439740 \\ 7329 \times 300 &= \underline{2198700} \\ \therefore 7329 \times 365 &= 2675085 \end{aligned}$$

But in practice as the cyphers arising from the higher places of the multiplier, are lost in the addition, we omit them.

ART. 9. A number which cannot be produced by the multiplication of two others, or cannot be separated into factors, is called a *prime* number, such as 3, 5, 7, 11.

A number which can be produced by the multiplication of two or more smaller ones, is called a *composite* number, as 27 (9×3), or 48 (6×8).

If therefore the multiplier be a *composite* number, we shall simplify the working, if we multiply by the component parts, as by 7 and 5, in Ex. 2; and this is especially the case in the multiplication of compound quantities, as in Ex. 5. It is immaterial which factor we multiply by first, but it is better to use that which will immediately simplify the working. Thus in multiplying £17. 3*s*. 8*d*. by 75 ($5 \times 5 \times 3$) if we multiply by the 3 first we shall get rid of the pence. Other artifices will suggest themselves, as in multiplying £2. 17*s*. 9*d*. by 67, or $8 \times 8 + 3$. So in multiplying £17. 4*s*. 5*d*. by 106, multiply by 12 and 9 and subtract twice.

N.B.—The proof of multiplication by the *casting out nines* will be given in the Algebra. Vide Art. 50.

DIVISION.

ART. 10. Ex. 1. Divide 30114 by 63. Ans. 478.

2. Divide 974932 by 365. Ans. $2671 \frac{17}{365}$.

3. Divide 378643 by 5200. Ans. $72 \frac{4243}{5200}$.

4. Divide 53872694 by 73. Ans. $737982 \frac{8}{73}$.

5. Divide 34568795 by 9879. Ans. $3499\frac{2174}{9879}$.

6. Divide £257. 2*s.* 3*d.* by 12. Ans. £21. 8*s.* $6\frac{1}{4}$*d.*

7. Divide £465. 12*s.* 8*d.* by 72. Ans. £6. 9*s.* $4\frac{1}{9}$*d.*

8. Divide 124 tons 9 cwt. by 38. Ans. 3 tons 5 cwt. 2 qr.

9. What is the eighteenth part of 21 acres 24 poles?
Ans. 1 acr. 28 pls.

10. Divide 3587 linear yds. 9 ins. by 27.
Ans. 132 yds. 2 ft. 7 ins.

ART. 11. Division is the subtraction of the same number several times, and is a concise method of performing the operation of many subtractions; in the same way as multiplication supplies the place of many additions. It consists in subtracting the divisor from the dividend, as often as it can be done, and reckoning the number of subtractions. And in the same manner as in multiplication, the multiplier shews how many *additions* are necessary to produce the number; so in division, the quotient shews how many *subtractions* are necessary to exhaust it.

These operations mutually prove each other. In multiplication, the product *divided* by the multiplier gives the multiplicand; in division, the quotient *multiplied* by the divisor gives the dividend; or we may prove it by casting out the nines in the divisor and quotient.

In division also the common method is an abbreviated one. Thus in the simple example of dividing 5936 by 8, we first enquire how many hundred times the divisor is contained in the dividend, and subtract the amount of these hundreds. Then, how often it is contained 10 times in the remainder, and subtract the amount of those tens, and lastly, how many single times it is contained in what remains after the last subtraction. But it being obvious that 8 will be con-

$$\begin{array}{r}
8)5936(700+40+2 \\
\underline{5600} \\
336 \\
\underline{320} \\
16 \\
\underline{16} \\
\underline{0}
\end{array}$$

tained in 59 as many *single* times as it will be contained *hundred* times in 5900 or 5936, and will be contained in 33 as many *single* times as it will be contained *ten* times in 330 or 336; therefore for the sake of convenience the cyphers are omitted in common practice, as in multiplication.

Again, since by the principles of Notation, every figure to the *left* of another has a *local* value ten times greater than that other, therefore in the division of *abstract* numbers, if there be a remainder, as 3 in dividing the 59 of the last example, this is multiplied by 10 and added to the next figure, which is 3, (making 33), then the 33 divided again, and the next remainder 1 is multiplied by 10 and added to the 6, and so on if there be more figures. A similar observation holds also in the division of *concrete* or *compound* numbers, as money, weight, &c. The pounds that remain after dividing by any number are multiplied by 20 to bring them into the next denomination (shillings), and the shillings that remain after division are multiplied by 12, and so on for the pence.

If there be a remainder after the division is completed, it is expressed by means of a vulgar fraction, whose numerator is the remainder, and denominator the divisor, as in Examples 2, 3, 4, 5.

If the divisor be a *composite* number it is sometimes more simple to divide by the factors, as by 9 and 7 in Example 1, or by 8 and 9 in Example 7, dividing by the 8 first because it leaves no remainder. So in dividing 975 by 105 ($5 \times 7 \times 3$), divide by the 7 last for a similar reason, and the result is $9\frac{2}{7}$.

If the divisor be any number with one or more cyphers to the right hand, as in Example 3, we may mark off an equal number of figures from the dividend, and annex them to the remainder. The reason of this is obvious from the use of cyphers in Notation, and will appear by performing the operation at large and comparing the steps.

ART. 12. In the foregoing examples in Division, we have divided a *compound* number as money, weight, &c., by a simple or *abstract* number; but it is manifest from Example 8, that if 124 tons 9 cwt., divided by 38 is 3 tons 5 cwt. 2 qrs., then 124 tons 9 cwt., divided by the compound quantity 3 ton 5 cwt. 2 qrs. is 38. The method of performing this will be given in Reduction.

We may remark, that though we are able to *divide* one compound quantity by another of the same kind, as money by money, weight by weight &c., yet we cannot generally *multiply* together two compound quantities whether of the same or different kind; though we are able to multiply feet by feet, as will be shewn in Duodecimals.

ART. 13. In order to determine the factors in any composite number, (in working examples in Multiplication, Division, &c.,) it may be useful to remember, that

(1) Every number is divisible by 2, if it ends in an even number or 0, as 34 or 70.

(2) Every number is divisible by 5, if it ends in 5 or 0, as 35, 80.

(3) Every number is divisible by 10, if it ends in 0, as 30.

(4) Every number is divisible by 4, if its last *two* digits are divisible by 4, as 732, since 32 is divisible by 4.

(5) Every number is divisible by 8, if its last *three* digits are divisible by 8, as 1592, since 592 is divisible by 8.

(6) Every number is divisible by 3 or 9, if the sum of its digits is divisible by 3 or 9 respectively. Thus 543 is divisible by 3, because the sum of its digits 12 is divisible by 3; and 828 is divisible by 9, because the sum of its digits 18 is divisible by 9.

REDUCTION.

Art. 14. Reduction is the operation by means of which quantities expressed in one denomination are converted into another, whilst their values remain the same (as from pounds to pence, from inches to yards, &c.) and is performed by multiplication and division. All quantities of a higher denomination are reduced to quantities of a lower by *multiplying* by so many of the lower as make one of the higher; and all quantities of a lower denomination are reduced to those of a higher by *dividing* by as many of the lower as make one of the higher.

Ex. 1. Reduce £8. to pence.

£8.
20
160 shillings
12
1920 pence

Here we have to reduce a quantity of a higher denomination to a lower. Now, since there are 20*s.* in £1., in any number of pounds there will be twenty times as many shillings, therefore we first multiply the 8 by 20 to bring them into shillings, and then since there are 12 pence in 1 shilling, we multiply the shillings by 12 to bring them into pence, and the answer is 1920.

Ex. 2. Reduce £46. 13*s.* $8\frac{3}{4}$*d.* to farthings.

£46. 13*s.* $8\frac{3}{4}$*d.*
20
933 shillings
12
11204 pence
4
44819 farthings

Here the given quantity consists of different denominations, and we add in with each product the term of the corresponding denomination, as the 13*s.* to the shillings, the 8*d.* to the pence, and the $\frac{3}{4}$ to the farthings, and the answer is 44819, or there are 44819 farthings in £46. 13*s.* $8\frac{3}{4}$*d.*

Ex. 3. Reduce £12. 5*s.* and £19. 10*s.* 6*d.* to the same denomination.

£12. 5*s.*	£19. 10*s.* 6*d.*
20	20
245 shillings	390 shillings
2	2
490 sixpences.	781 sixpences.

N.B. It is not necessary to reduce them to a lower denomination than sixpences, if we have only to reduce them to the *same* denomination.

Ex. 4. In 27 ounces of gold, how many grains?

```
  27 ounces
   20
  540 dwts.
   24
 2160
1080
12960 grs.
```

Here we multiply first by 20 because 20 dwts. make 1 oz., and secondly by 24 because 24 grs. make 1 dwt., and the answer is 12960.

Ex. 5. In 34 ton 17 cwt. 1 qr. 19 lb. how many pounds?

```
  34 ton  17 cwt.  1 qr.  19 lb.
  20
  697 cwt.
    4
 2789 qrs.
   28
22331
5578
78111 lb.
```

Here we multiply by 20, because 20 cwt. make 1 ton, adding in the 17 cwt.; then by 4, because 4 qrs. make 1 cwt., adding in the 1 qr., and similarly for the pounds, and the answer is 78111.

Ex. 6. In 72 leagues how many furlongs and yards?

```
   72 leagues
    3
  216 miles
    8
 1728 furls.
  220
34560
3456
380160 yds.
```

Here we multiply by 3 to bring them into miles, by 8 into furlongs, and by 220 (or $5\frac{1}{2} \times 40$) because 220 yards make 1 furlong, and the answer is 380160 yards.

Ex. 7. Reduce 7680 farthings to pounds.

```
 4)7680 f.
12)1920 d.
20) 160 s.
     £8.
```

Here we have to reduce a quantity of a lower denomination to a higher. Now, since there are 4 farthings in 1 penny, in any number of farthings there will be $\frac{1}{4}$th times

as many pence, therefore we first divide the farthings by 4 to bring them into pence, and then since there are 12 pence in 1 shilling, we divide the pence by 12 to bring them into shillings, and so on for the pounds, and the answer is £8.

Ex. 8. Reduce 44819 farthings to pounds.

```
 4)44819
12)11204. 3f.
  20)933. 8d.
    £46. 13s. 8¾d.
```

Here there is a remainder after all the divisions, and we set it down as a term of the same denomination as the dividend from which it came. Thus, after dividing by 4 we set down the remaining 3 as 3 farthings, because the dividend is farthings, and the remainder 8 as 8*d.*, because the dividend is pence, and so on for the pounds, and the answer is £46. 13*s.* 8¾*d.*

Ex. 9. In 4320 perches how many acres?

```
40)4320 sq. perches
  4)108 roods
     27 acres
```

Here we divide by 40, because 40 poles make 1 rood, and by 4, because 4 roods make 1 acre.

Ex. 10. Reduce 174865 grains to pounds.

```
   ⎧8)174865 grs.
24 ⎨
   ⎩ 3)21858. 1 grain
     20)7286 dwts.
       12)364 oz. 6 dwts.
          30. 4. 6. 1
```

Here we divide by 24 (3×8) because 24 grains make 1 dwt., and the remainder 1 is 1 grain, because the dividend is in grains; and then by 20 and 12, because 20 dwts. make an ounce and 12 ounces, 1 pound.

Ex. 11. Reduce 3427 half-crowns to pounds.

```
8)3427 half-crowns
  £428. 3 half-crowns, or £428. 7s. 6d.
```

We divide by 8, because 8 half-crowns make £1., and the remainder is 3 half-crowns, or 7*s.* 6*d.*

Ex. 12. In 72015 hours, how many weeks?

$$\begin{array}{l} 24\underline{)72015} \text{ hours} \\ \quad 7\underline{)3000} \text{ days } 15 \text{ hours} \\ \qquad \underline{428} \text{ wks. } 4 \text{ days } 15 \text{ hrs.} \end{array} \qquad 24\left\{\begin{array}{l} 3\underline{)72015} \\ 8\underline{)24005} \\ \quad \underline{3000.}\ 5 \end{array}\right.$$

Here we divide by 24, because 24 hours make 1 day, &c., and the answer is 428 weeks 4 days 15 hours.

N. B.—Great care must be taken, if we divide by the component factors of a number, as 24 or 8 × 3, to give the remainder (if there be any) its proper value. Thus it will be seen in the above example (second case) that when we divide by 3 there is no remainder, but when this quotient is again divided by 8 the remainder is 5, not 5 *hours*, but 5 × 3 hours, or 15 hours (which agrees with case 1) because, if we may so speak, when we divide by 3 we reduce them into THREE-HOURS, and the remainder 5, which must be of the same denomination as the dividend, will be 5 *three-hours* or 15 hours. In Example 10 the case was different, because there was no *second* remainder, and the first remainder was of course of the same denomination as the dividend, *i. e.* grains.

Ex. 13. Reduce £31742. to guineas.

$$\begin{array}{l} \quad £31742 \\ \qquad\quad \underline{20} \\ 21\left\{\begin{array}{l} 7\underline{)634840} \text{ shillings} \\ \ \ 3\underline{)90691.}\ 3 \text{ shillings} \end{array}\right. \\ \qquad\quad 30230 \text{ guineas } 1, \text{ or equal to } 30230 \text{ gs. } 10 \text{ sh.} \end{array}$$

This is an example of *Mixed Reduction.* There is no exact number of pounds in a guinea, and therefore we reduce the pounds by multiplication into shillings, and then since 21 shillings make 1 guinea, we reduce the shillings by division to the required denomination. The same remark applies here as in the last Example. The 1 which remains after the second division is 1 *seven shillings,* and added to the former 3 shillings makes 10 shillings. There will be less chance of making an error, if we divide by 21 at once.

Ex. 14. Reduce 325 crowns to half-guineas.

325 crowns
10
21)3250 sixpences
154 half-guineas 16 sixpences, or 154 half-guineas 8*s*.

Here we multiply by 10 to bring them into sixpences—a common denomination to crowns and half-guineas—and then divide by 21, because 21 sixpences make half-a-guinea, the remainder 16 being in *sixpences*, the same denomination as the dividend.

Ex. 15. Reduce £11. 7*s*. $9\frac{3}{4}$*d*. into pieces of money worth £1. 5*s*. $3\frac{3}{4}$*d*.

£11. 7*s*. $9\frac{3}{4}$*d*.	£1. 5*s*. $3\frac{3}{4}$*d*.
20	20
227 shillings	25 shillings
12	12
2733 pence	303 pence
4	4
10935 farthings	1215 farthings

1215)10935(9
10935
.....

This is an Example of the class alluded to at the end of Division, viz. the method of dividing a *compound quantity* by a *compound* quantity. We first reduce them to the same denomination, *i. e.* farthings, and then to the required denomination by dividing by as many farthings as make a unit of that denomination.

Ex. 16. A piece of iron weighs 3 cwt. 2 qrs. 8 lb., how many such pieces will be contained in 96 cwt. 1 qr. 20 lb.?

3 cwt. 2 qrs. 8 lb.	96 cwt. 1 qr. 20 lb.
4	4
14 qrs.	385 qrs.
28	28
120	3100
28	770
400 lbs.	10800 lbs.

4,00)108,00
27

Here we reduce them to the same denomination, lbs, and then divide as before.

Ex. 17. Reduce £37. to pence. Ans. 8880.

Ex. 18. Reduce £31. 5*s*. 6½*d*. to half-pence. Ans. 15013.

Ex. 19. Reduce £17. 5*s*. 3¼*d*. to farthings. Ans. 16573.

Ex. 20. In 15 crowns, how many shillings and sixpences? Ans. 75*s*. 150 sixpences.

Ex. 21. Reduce 21 guineas to farthings. Ans. 21168.

Ex. 22. Reduce 125 yds. 2 ft. 4 in. to inches. Ans. 4528 in.

Ex. 23. Reduce 1 mile 13 poles to inches. Ans. 65934 in.

Ex. 24. How many minutes are there in a year, which consists of 365¼ days? Ans. 525960.

Ex. 25. How many seconds are there in 57 degrees 17 minutes 44 seconds? (Vid. Table 8.) Ans. 206264.

Ex. 26. How many square feet are there in an acre? Ans. 43560.

Ex. 27. Reduce 3 lb. 10 oz. 7 dwt. 5 grs. to grains. Ans. 22253.

Ex. 28. Reduce 9 cwt. 2 qrs. 14 lb. to pounds. Ans. 1078.

Ex. 29. Reduce 15783 pence to pounds. Ans. £65. 15*s*. 3*d*.

Ex. 30. Reduce 16573 farthings to pounds. Ans. £17. 5*s*. 3¼*d*.

Ex. 31. Reduce 588483 half-pence to guineas. Ans. 1167*gs*. 13*s*. 1½*d*.

Ex. 32. Reduce 14769 oz. to cwt. Ans. 8 cwt. 27 lb. 1 oz.

Ex. 33. Reduce 18191 pints to gallons. Ans. 2273 gals. 3 qts. 1 pt.

Ex. 34. In 108 barrels of beer, how many hogsheads.
Ans. 72.

Ex. 35. Reduce 76 moidores into pounds. Ans. £102. 12*s*.

Ex. 36. Reduce 253 guineas to crowns. Ans. 1062 and 3*s*.

Ex. 37. In 27 cwt., how many parcels of 18lb. each.
Ans. 168.

Ex. 38. In £100., how many piastres at 3*s*. 7*d*. each.
Ans. $558\frac{6}{43}$.

Ex. 39. Divide £45. 14*s*. $0\frac{3}{4}d$. by 2*s*. $7\frac{1}{4}d$. Ans. 351.

Ex. 40. How many coins, each worth 4*s*. 9*d*., are there in £231. 16*s*. Ans. 976.

Ex. 41. What is the price of one yard, when 48 cost £15. 10*s*. 4*d*. Ans. 6*s*. $5\frac{1}{2}d.\frac{1}{3}$

Ex. 42. How many dollars at 4*s*. 2*d*. are there in £20.
Ans. 96.

Ex. 43. Divide 244 qrs. 3 bush. 1 peck, by 3 qrs. 3 pecks.
Ans. 79.

Ex. 44. In £58. 12*s*. 6*d*., how many francs at 10*d*. each.
Ans. 1407.

Ex. 45. How many pieces of coin of 7*s*. 2*d*., are there in £111. 16*s*. Ans. 312.

Ex. 46. How many pieces of coin of 1*s*. 7*d*. each, are there in £2827. 4*s*. Ans. 35712.

RULE OF THREE.

Art. 15. The Rule of Three, or as it is commonly called Proportion, is a method of finding, when three numbers are given, a fourth to which the third may have the same proportion that the first has to the second.

It consists of two parts,

(1) The single Rule of Three, or Simple Proportion.

(2) The double Rule of Three, or Compound Proportion.

Generally four numbers are in the same proportion, when the first contains the second or any part of it, as often as the third contains the fourth, or any part of it. In either of these cases, the quotient of the first divided by the second equals the quotient of the third divided by the fourth, and this quotient may be called the measure of the proportion.

When four numbers are proportional to one another, as 2, 4, 8, 16, they are written thus,

$$2 : 4 :: 8 : 16$$

$$\text{or } 2 : 4 = 8 : 16$$

which signifies that 2 has the same proportion to 4, that 8 has to 16; and because the quotient of the first divided by the second equals the quotient of the third divided by the fourth, this proportion is sometimes expressed as a fraction, thus

$$\frac{2}{4} = \frac{8}{16}.$$

We may remark from above, that the fourth term 16 is the product of the second and third terms, *i. e.* 4 and 8 divided by the first term 2. Hence generally, if we have three numbers given in proportion, we may find the fourth by multiplying the second and third terms together, and dividing by the first. For example, find a fourth proportional to 18, 27 and 34.

This will be the product of 27 and 34 divided by 18, or 918 divided by 18, which is 51, and therefore

$$18 : 27 = 34 : 51.$$

To explain the reason of this Rule we must observe, that if two or more numbers be multiplied or divided alike, the products or quotients will have the same proportion.

Thus		18 : 27
	multiplying by 34,	612 : 918
	dividing by 18,	34 : 51

the products 612, 918, and the quotients 34, 51, have the same proportion to each other that 18 has to 27. And in this operation the product of the first and third terms are divided by

the first, and therefore the quotient 34 must be the third, and 51 to which it is proportional the fourth, and we see that this fourth term 51 is obtained by multiplying the second and third terms together, and dividing by the first, as appears from the second column.

In the proportion of compound or concrete quantities the first and second terms must always be of the same kind as *both* money, *both* weight, &c. &c., and the fourth will be of the same kind as the third. For example,

$$5 : 7 = £25 : £35.$$

not $5 : £25 = 7 : £35.$, since there can be no relation between *abstract* and *compound* numbers.

ART. 16. Having made these few observations on the nature of proportion, we proceed to the Single Rule of Three, or Simple Proportion, in which we have to find a fourth proportional to three given quantities. Instead however, of giving Rules for *stating* examples in this operation, we have considered it more simple to work them by the method which common sense suggests; and this method will be found much preferable in Compound Proportion.

Now we may remark that the value, weight, and measure of any commodity will increase as the quantity increases, that the amount of work done in any time will increase if the time be increased, and generally the effect produced by any cause will be proportional to the extent of the cause. On the other hand, when the price of goods increases, the quantity which may be bought for a given sum is smaller; when the number of men employed at work is increased, the time in which they may complete it becomes shorter; and when the activity of any cause is increased, the quantity necessary to produce a given effect is diminished. In the former of these cases the proportion is called *direct*, in the latter *inverse*. For example, "If 30 horses plough 12 acres, how many will 42 plough in the same time." Here the number of acres will be greater because the number of horses is greater, and the proportion *direct*.

Again, "If 40 horses be maintained for a certain sum on hay at $5d.$ per stone, how many will be maintained on the same sum when the price rises to $8d$?" Here the number of horses will be less because the price of hay increases, and the proportion *inverse*.

Every question of this kind consists of a supposition and a demand, as

Ex. 1. If 1 lb. of sugar cost $4\frac{1}{2}d.$, what will 54 lb. cost?

1 pound of sugar costs $4\frac{1}{2}d.$
therefore 2 pounds .... $9d.$
multiplying by 27, 54 pounds .... $9 \times 27d.$ or
$243d. = £1.\ 0s.\ 3d.$

The working of this example is self-evident (without *stating* it as a proportion) and we might at once have multiplied by 54.

Ex. 2. If 36 yards cost $42s.$ what will 27 cost?

36 yards cost $42s.$
therefore dividing by 4, 9 yards .. $10s.\ 6d.$
multiplying by 3, 27 yards .. $31s.\ 6d.$, or
36 yards : 27 yards = $42s.$: $31s.\ 6d.$

In this Example the value of 36 yards is given, and we are required to find that of 27 yards. Now the factors of 27 being 9 and 3, if we can find the value of 9 or 3 yards, we can immediately obtain that of 27 yards. In finding the factors or component parts of numbers, attention should be paid to the properties of numbers mentioned in Art. 13. If however this method be thought difficult, we can always obtain at once by division the value of a *unit* of the required kind, and then by multiplication the value of any number of units. Thus

36 yards cost $42s.$
therefore 1 .... $1s.\ 2d.$
therefore 27 .... $31s.\ 6d.$

Ex. 3. If £57. will buy 19 cwt. of sugar, what quantity may be had for £111.?

£57. will buy 19 cwt.
therefore (dividing by 19) £3. ...... 1 cwt.
therefore (multiplying by 37) £111 ...... 37 cwt.
or £57 : £111 = 19 cwt. : 37 cwt.

Here 111 = 3 × 37, and the purchase of £3. is immediately obtained; or it might stand thus,

£57. will buy 19 cwt.
therefore £1. .... $\frac{19}{57}$ of a cwt.
therefore £111. ...... 111 × $\frac{19}{57}$ or 3 × 19 = 37 cwt.

Ex. 4. If 32 yards cost £24. 16*s*., what is the value of a yard?

32 yards cost £24. 16*s*.
therefore (dividing by 8) 4 yards .. £3. 2*s*.
therefore 1 yard .. 15*s*. 6*d*.

Ex. 5. If the pendulum of a clock vibrate 240 times in 4 minutes, how many times will it vibrate in $1\frac{3}{4}$ hours.

In this as in other Examples where there are different denominations, as lbs, ozs.; pounds, shillings; bushels, pecks, &c., &c., first bring them by Reduction into the same denomination.

Now by Reduction $1\frac{3}{4}$ hours = 105 minutes
By the question, the pendulum vibrates 240 times in 4 minutes.
therefore .... 60 ...... 1 minute.
.... 60 × 105 .... 105 minutes.
or it vibrates 6300 times in $1\frac{3}{4}$ hours.

Ex. 6. If 7 ounces 11 dwt. of gold be worth £35. what is the value of 14 lb. 9 oz. 12 dwts. 16 grs. at the same rate?

By Reduction 14 lb. 9 oz. 12 dwts. 16 grs. = 85264 grains.
.... 7 oz. 11 dwts. = 3624 grains
Now 3624 grains cost £35.
therefore (dividing by 2) 1812 .... £17. 10*s*.
.... 906 .... £8. 15*s*., or 175*s*.
therefore 1 grain costs $\frac{175}{906}$ shillings,
therefore 85624 grains costs 85624 × $\frac{175}{906}$ shillings,

or by working out the multiplication and division,

14 lb. 9 oz. 12 dwts. 16 grs. cost £823. 9*s*. $3\frac{3}{4}d$.

Ex. 7. If 24 lb. cost 6*s*. 6*d*. what will 18 parcels cost, each weighing 3 qrs. 18 lb.

By reduction 3 qrs. 18 lb. = 102 lb.

Now 24 lb. cost 6*s*. 6*d*.;

∴ 12 lb. .... 3*s*. 3*d*., or 39*d*.

∴ 2 lb. .... $\frac{39}{6}\,d.$

∴ 102 (= 2 × 51) .... or $\frac{51 \times 39}{6}\,d.$

∴ the 18 parcels cost ...... $\frac{18 \times 51 \times 39}{6}$ or $3 \times 51 \times 39d.$

or £24. 17*s*. 3*d*.

In all examples of this sort, the reduction, and the multiplication and division must be *fully* worked out.

Obs. In the above examples the proportion has been *direct;* the following examples are in *inverse* proportion.

Ex. 8. If 8 men can do a piece of work in 12 days, in how many days can 16 men perform the same?

8 men do the work in 12 days;

therefore twice as many men, or 16 men will do the same work in half as many days, or in 6 days.

Here it is evident, that the number of days will be diminished in proportion as the number of men are increased, or the number of days varies *inversely* as the number of men.

In the former examples we multiplied or divided both terms by the same number, in these if we *multiply* one term by any number we *divide* the other by the same number, or *vice versâ*.

Ex. 9. If 108 men finish a piece of work in 12 days; how many are sufficient to finish it in 3 days?

108 men do the work in 12 days

∴ (multiplying and dividing by 12) 1296 men ...... 1 day.

∴ (dividing and multiplying by 3) 432 men ...... 3 days.

Ex. 10. If a besieged garrison have provisions for 21 days, at the rate of 20 ounces a day for each man, to what quantity must each man's allowance be reduced, in order that the provisions may last 30 days.

The provisions will last 21 days at 20 ounces per man per day
∴ (dividing and multiplying by 7) .... 3 days at 140 ......
∴ (multiplying and dividing by 10).... 30 days at 14 ......
or each man's allowance will be 14 ounces per day.

Obs. All these examples can be worked by the common method if it be thought preferable, the rule for which is,

Put that term last which corresponds to the answer, and the greatest or least of the other 2 terms second, according as the answer is to be greater or less than the third term. Then reduce the first and second terms to the *same* denomination, and the third into any denomination we please, and multiply together the second and third terms, and divide the product by the first. The quotient will be the answer in the same denomination as that in which we left the third term.

Ex. 11. If 3 yards cost 15*s*. 9*d*. what will 7 cost?
Ans. £1. 16*s*. 9*d*.

Ex. 12. If 1 lb. of sugar cost $10\frac{1}{2}$*d*. how many may be bought for £4. 18*s*. Ans. 1 cwt.

Ex. 13. If a man's annual income be £408. 16*s*., how much does he receive for 15 days? Ans. £16. 16*s*.

Ex. 14. What weight ought to be carried $25\frac{3}{4}$ miles for the same sum for which 3 cwt. are carried 40 miles?
Ans. 4 cwt. 2 qrs. $17\frac{97}{103}$ lbs.

Ex. 15. If $17\frac{3}{5}$ ells, each containing 5 quarters, cost £6. 17*s*. how much will 18 yards cost? Ans. £5. 12*s*. $1\frac{1}{11}$*d*.

Ex. 16. If 3 cwt. 3 qrs. cost £6. 15*s*. what will be the price of 2 cwt.? Ans. £3. 12*s*.

Ex. 17. If 3 cwt. 3 qrs. 27 lbs. cost £5. 16*s.* what will be the price of 5 cwt. 2 qrs. at the same rate?

Ans. £7. 19*s.* 10$\frac{1}{4}$*d.* $\frac{19}{149}$.

Ex. 18. If 5 pieces of cloth, each containing 17$\frac{1}{2}$ yards, cost £19. 13*s.* 9*d.* what will a piece 24 yards long cost?

Ans. £5. 8*s.*

Ex. 19. The carriage of a parcel of goods weighing 1 ton. 3 cwt. 2 qrs. cost £2. 14*s.* what will be the charge for 4 other parcels each weighing 17 cwt. 3 qrs. 7 lbs.? Ans. £8. 3*s.* 8$\frac{1}{2}$*d.* $\frac{34}{47}$.

Ex. 20. If 54 men can build a house in 90 days, how many men can do the same in 50 days? Ans. 97$\frac{1}{5}$.

Ex. 21. If for a given sum I have 1200 lb. carried 36 miles, how many pounds can I have carried 24 miles for the same sum? Ans. 1800 lb.

Ex. 22. How much land at 27*s.* an acre should be given in exchange for 480 acres at 35*s.* an acre? Ans. 622$\frac{2}{9}$ acres.

Ex. 23. In what time will 25 men do a piece of work which 12 men can do in 3 days? Ans. 1$\frac{11}{25}$ days.

Ex. 24. If a man walk 62 miles in 3 days, in how many days will he walk 80 miles? Ans. 3$\frac{27}{31}$ days.

Ex. 25. If a man walking 7 hours a day, finish his journey in 9 days, in how many days would he have finished it if he had walked 10 hours a day. Ans. 6$\frac{3}{10}$ days.

Ex. 26. If 20 men can perform a piece of work in 12 days, how many men can perform a piece of work 3 times as great in one fifth part of the time. Ans. 300.

Obs. Here, though there are 4 terms, they are reducible to 3, since it is evident that 60 men will do the work of the *required* kind in 12 days, and then we have only to find how many will perform the *same* in $\frac{1}{5}$th of 12 days.

Ex. 27. A man sells an article for 5*s*. and gains 20 per cent, what was its prime cost?

Since he gains 20 per cent on the article *sold*,
therefore what he sells for 120*d*. cost him 100*d*.,
therefore .... 60*d*. 50*d*.,
therefore what he sells for 5*s*. costs him 4*s*. 2*d*.

Ex. 28. If 25 yards of butter cost 30*s*., what must it be sold at per foot to gain 5*s*. by the whole. Ans. $5\frac{3}{5}d$.

Ex. 29. A person whose debts are £1500, pays £950., how much is that in the pound. Ans. 12*s*. 8*d*.

Ex. 30. What is the income corresponding to an income-tax of £13. 2*s*. 6*d*., at the rate of 7*d*. in the pound. Ans. £450.

COMPOUND PROPORTION,

OR DOUBLE RULE OF THREE.

Art. 17. The difference between Simple and Compound Proportion is, that in the former, the Proportion depends on *one* circumstance only, in the latter, it depends upon *several* circumstances, and therefore requires a repetition of the process made use of in the former. And because the questions occurring in Compound Proportion frequently involve two Simple Proportions, it is also called the *Double* Rule of Three. Thus, "if 18 men consume 6 qrs. of corn in 28 days, how much will 24 men consume in 56 days."

Here the quantity required depends *partly* on the number of men, and *partly* on the time, and the question may therefore be resolved in two Simple Proportions, as follows:

(1) If a number of men consume 6 qrs. of corn in 28 days, how much will they consume in 56 days. Ans. 12qrs.

(2) If 18 men consume 12 qrs. in any time, how much will 24 consume in the same time. Ans. 16 qrs.

In this Example there are five terms given, and we have to

find a sixth; in Simple Proportion there are only three terms given to find a fourth. Every question which falls under the head of Compound Proportion *must* have five terms given, and may have more as in Example 5, below.

A careful consideration of the sense of the question will, in this case, also as in the former, enable us to discover whether the Proportion is *direct* or *inverse*.

Ex. 1. If 15 oxen can eat 1 acre of grass in 12 days, how long will it take 26 oxen to eat 14 acres.

15 oxen eat 1 acre of grass in 12 days.

∴ .... 14 .... 12×14 or 168 days (direct)

∴ 1 ox .... 15×168 days (inverse)

∴ 26 oxen .... $\dfrac{15 \times 168}{26} = 96\frac{12}{13}$ days.

The steps in this Example are self-evident, and far more intelligible than the *Rules* generally given for the solution of examples of this kind.

Ex. 2. If 100 men make 3 miles of road in 27 days, in how many days will 150 men make 5 miles.

100 men make 3 miles in 27 days

∴ 50 .... 54 days (inverse)

∴ 150 .... 18 days ......

∴ 150 .... 1 mile .. 6 days (direct)

∴ 150 .... 5 ...... 30 days. ....

Ex. 3. If 2 horses eat 8 bushels of oats in 16 days, how many horses will eat 3000 qrs. in 24 days?

2 horses eat 8 bushels or 1 qr. in 16 days

∴ 4 .... 1 qr. in 8 days (inverse)

∴ 4 .... 3 qrs. in 24 days (direct)

∴ 4000 .... 3000 qrs. in 24 days.

Ex. 4. If the carriage of 60 cwt. 20 miles cost £14. 10*s.* what weight can I have carried 30 miles for £5. 8*s.* 9*d.*?

By reduction £5. 8*s.* 9*d.* = 1305*d.* £14. 10*s.* = 290*s.*

Now 290*s*. is the carriage of 60 cwt. 20 miles,

$\therefore$ 29*s*. . . . . 6 . . . . 20 . . . . (direct)

$\therefore$ $\frac{29}{20}s$. . . . . 6 . . . . 1 . . . . (direct)

$\therefore$ $30 \times \frac{29}{20}s$. or 522*d*. . . . . . 6 . . . . 30 . . . .

$\therefore$ 87*d*. is the carriage of 1 cwt. for 30 miles, dividing by 6,

$\therefore$ 1305*d*. or 87×15 is the carriage of 15 cwt. for 30 miles,

or 15 cwt can be carried 30 miles for £5. 8*s*. 9*d*.

Ex. 5. If 18 men consume £30. in value of corn, in 9 months, when the price is 16*s*. per quarter, how many will consume £54. in value in 6 months, when the price is 12*s*. per qr.

Here the number of men depends on three things,

(1) the value of the corn, (2) the time, (3) the price per qr.

The first of these is *direct*, because the more the value of provisions is, the more time is required to consume them; the second and third *inverse*, for the greater time and price is, the fewer will be the number of men to consume an equal value.

18 men consume £30. of corn in 9 months when it is 16*s*. per qr.

$\therefore$ . . . . £10. . . . . . . 3 . . . .

$\therefore$ . . . . £20. . . . . . . 6 . . . .

$\therefore$ . . . . £5. . . . . 4*s*. per qr.

$\therefore$ . . . . £15. . . . . 12*s*. per qr.

$\therefore$ $\frac{54 \times 18}{15}$ men £54. . . . .

or $64\frac{4}{5}$ men will consume £54 in value, when corn is 12*s*. per qr.

Ex. 6. If 112 lb. avoirdupois, make 104 lb. of Holland, and 100 lb. Holland make 89 lb. of Geneva, and 110 lb. of Geneva make 117 lb. of Seville, how many pounds Seville will make 100 lb. avoirdupois?

112 lb. av. make 104 of Holland,

$\therefore$ 112×100 av. . . . . 104×100 . . . . . .

or 104×89 Gen. (since 100 Hol. = 89 Gen.)

$\therefore$ $112 \times 100 \times 110$ av. make $104 \times 89 \times 110$ Gen.,

or $104 \times 89 \times 117$ Seville (since 110 Geneva = 117 Seville)

therefore 100 lbs. avoirdupois make $\frac{104 \times 89 \times 117}{112 \times 110}$ Seville.

If it be required on the same supposition how many pounds avoirdupois make 100 pounds Seville, then from above

$112 \times 100 \times 110 \times 100$ av. make $104 \times 89 \times 117 \times 100$ Seville,

or 100 lbs. Seville make $\frac{112 \times 100 \times 110 \times 100}{104 \times 89 \times 117}$ avoirdupois.

In a similar manner Examples in what is termed Distributive proportion may be done, as

Ex. 7. A bankrupt owes *A* £146., *B* £170., *C* £45., *D* £480., *E* £72., and his whole effects are £342. 7*s*. 6*d*.; how much should each have?

The sum of these several debts is £913. or 7304 half-crowns, and his whole effects are £342. 7*s*. 6*d*., or 2739 half-crowns.

Taking *D*'s case only, he owes him 3840 half-crowns, and the question is, if he pay 2739 half-crowns instead of 7304 half-crowns, what will he pay instead of 3840 half-crowns.

Now he pays 2739 instead of 7304 half-crowns.

$\therefore$ (dividing by 913,) he pays 3 instead of 8 half-crowns.

$\therefore$ he pays 480×3 instead of 480×8 half-crowns,

or 1440 instead of 3840 half-crowns;

or he pays *D* £180. instead of £480.

Similarly he pays *A* £54. 15*s*.; *B* £63. 15*s*.; *C* £16. 17*s*. 6*d*.; *E* £27.; the sum of which payments equal £342. 7*s*. 6*d*.

It will be seen that he pays 7*s*. 6*d*. in the pound.

In the above Example we have divided by 913 at once, because it is exactly contained in both terms; the more direct method would have been as follows:

he pays 2739 instead of 7304 half-crowns,

$\therefore$ ...... $\frac{2739}{7304}$ ...... 1 half-crown,

$\therefore$ ...... $\frac{3840 \times 2739}{7304}$ instead of 3840 half-crowns.

And in all examples of this sort if it be difficult to split the number into its component factors, as 2739 and 7304 into 913×3 and 913×8, this last method can always be adopted.

All these examples can be worked by the common method, if it be preferred, the rule for which is exactly the same as in simple proportion, the last term being the one that corresponds to the answer.

Ex. 8. If 7 men can reap 6 acres in 12 hours, how many men will reap 15 acres in 14 hours? Ans. 15.

Ex. 9. If 800 soldiers consume 3 barrels of flour in 6 days, how many will consume 15 barrels in 2 days? Ans. 12000.

Ex. 10. If a family of 9 persons spend £300. in 8 months, how much money will serve 17 persons 11 months, at the same rate of expenditure? Ans. £779. 3*s.* 4*d.*

Ex. 11. If 8 ounces of bread are sold for 6*d.* when wheat is £15. a load, what should be the price of wheat when 12 oz. are sold for 4*d.*? Ans. £6. 13*s.* 4*d.* per load.

Ex. 12. A man can reap $345\frac{5}{7}$ square yards in an hour; how long will 7 such labourers take to reap a field of 6 acres?

Ans. 12 hours.

Ex. 13. If two men in a tour of 3 months spend £140., how much at the same rate would it cost a party of 5 persons, travelling 11 months? Ans. £1283. 6*s.* 8*d.*

Ex. 14. If the carriage of 2 tons for 6 miles be 10*s.*, what will be the carriage of 12 tons 17 cwt. for 17 miles?

Ans. £9. 2*s.* $0\frac{1}{2}$*d.*

Ex. 15. If a regiment of 136 men consume 351 quarters of wheat in 108 days; how many quarters will 11232 soldiers consume in 56 days? Ans. $15031\frac{1}{17}$.

Ex. 16. If 5 men receive £18. 15*s.* wages for 12 months, what will be the wages of 16 men for 20 months? Ans. £100.

Ex. 17. If 36 oxen eat 216 acres in 1 year, and a sheep eat half as much as an ox, how long will 45 oxen and 136 sheep be eating 17628 acres together? Ans. 26 years.

Ex. 18. If 6 horses in 2 days of 12 hours each plough 17 acres, how many roods will 2 horses plough in 16 days of 8 hours each? Ans. $7\frac{5}{9}$ roods.

Ex. 19. If 12 men build 24 roods of wall in 30 days, working 8 hours a day, how many hours per day must 18 men work to build 72 roods in 40 days? Ans. 12.

Ex. 20. If 14 men, working 9 hours a day, dig a trench 140 yards long, 2 wide, and 2 deep, in 4 days; in how many days will 36 men, working 8 hours a day, dig a trench 280 yards long, 4 wide, and 2 deep? Ans. 7.

Ex. 21. Four partners engage to trade in company; *A*'s stock is £150., *B*'s £320., *C*'s £350., *D*'s £500., and they gain £730., how much belongs to each if the gain be divided among them in proportion to their stocks?

Ans. *A*'s £82. 19*s*. 1*d*.; *B*'s £176. 19*s*. 4*d*.
C's £193. 11*s*. 2*d*.; *D*'s £276. 10*s*. 3*d*.

Ex. 22. If 248 men in 5 days of 11 hours each can dig a trench 230 yards long, 3 wide, and 2 deep; in how many days of 9 hours each, can 24 men dig a trench 420 yards long, 5 wide, and 3 deep. Ans. $288\,\frac{59}{207}$.

Ex. 23. If 10 cannon, which fire 3 rounds in 5 minutes, kill 270 men in an hour and a half; how many cannon, which fire 5 rounds in 6 minutes, will kill 500 men in one hour at the same rate? Ans. 20.

Ex. 24. If 15 men, 12 women, and 9 boys, can complete a piece of work in 50 days; how long would 9 men, 15 women, and 18 boys be in doing double the quantity, the parts done by each in the same time being as the numbers 3, 2, 1?

Ans. 104 days.

CHAPTER II.

VULGAR FRACTIONS.

ART. 18. IF we suppose unity, or the number 1, to be divided into several equal parts, then one or more of these parts is called a *fraction*.

A *fraction* in its simplest form is represented by means of two numbers placed one above the other with a line between them. Thus two-fifth parts is written $\frac{2}{5}$.

The number under the line shews how many parts unity is divided into, and is called the *denominator*; the number above the line is called the *numerator*, and shews how many of those parts are taken to form the fraction. In the fraction $\frac{2}{5}$, 2 is the numerator and 5 the denominator; the 5 shewing that unity is divided into 5 equal parts, the 2 that 2 only of them are taken to form the fraction.

When the numerator is less than the denominator, the fraction is called a *proper fraction*, as $\frac{2}{5}$, $\frac{5}{8}$, &c. Hence a proper fraction is always less than unity.

When the numerator is equal to or greater than the denominator, the fraction is called an *improper fraction*, as $\frac{4}{4}$, $\frac{5}{2}$, $\frac{8}{5}$, &c. Hence an improper fraction is never less than unity.

When a fraction is annexed to an integer or whole number it is called a *mixed number*, as $5\frac{2}{3}$, which represents 5 units together with 2 third-parts of a unit.

A *compound fraction* is a fraction of a fraction, as $\frac{1}{2}$ of $\frac{2}{3}$, $\frac{5}{6}$ of $5\frac{2}{3}$, &c.

A *complex fraction* is one in which either the numerator or denominator or both, are fractions, as $\frac{5\frac{2}{3}}{2}$, $\frac{3}{1\frac{1}{2}}$, $\frac{5\frac{2}{3}}{1\frac{1}{2}}$ &c.

Every integer or whole number may be considered as a fraction whose denominator is 1. Thus 5 represented as a fraction is $\frac{5}{1}$.

Since division is sometimes denoted by placing the quantities like a fraction (Art. 2.); therefore every fraction may be considered as expressing the division of the numerator by the denominator.

ART. 19. To multiply a fraction by any integer.

Multiply the numerator of the fraction by the integer, and retain the same denominator; or divide the denominator of the fraction by the integer, and retain the same numerator.

Ex. Multiply $\frac{2}{10}$ by 2

$$\frac{2}{10} \times 2 = \frac{4}{10}, \text{ or } \frac{2}{10} \times 2 = \frac{2}{5}.$$

In the first case, the unit in each of the fractions $\frac{2}{10}$ and $\frac{4}{10}$ is divided into 10 equal parts, and twice as many of those parts are taken in the latter fraction as in the former. In the second case, the unit in the fraction $\frac{2}{10}$ being divided into twice as many equal parts as it is in the fraction $\frac{2}{5}$, each of the parts in the latter is twice as great as in the former, and the same number of parts being taken in both, the latter fraction is therefore twice as great as the former.

ART. 20. To divide a fraction by any integer.

Multiply the denominator of the fraction by the integer, and retain the same numerator; or divide the numerator of the fraction by the integer, and retain the same denominator.

Ex. Divide $\frac{2}{10}$ by 2.

$$\frac{2}{10} \div 2 = \frac{2}{20}, \text{ or } \frac{2}{10} \div 2 = \frac{1}{10}.$$

In the first case, the unit in the fraction $\frac{2}{20}$ being divided into twice as many equal parts as in the fraction $\frac{2}{10}$, each of the parts in the latter is twice as great as in the former, and the same number of parts being taken in both, the former fraction

is therefore one half of the latter. In the second case, the unit in each of the fractions $\frac{2}{10}$ and $\frac{1}{10}$ is divided into 10 equal parts, and half as many of those parts are taken in the latter fraction as in the former.

Ex. 1. Multiply $\frac{5}{16}$ by 7, 8, 9, 10.

Ans. $\frac{35}{16}, \frac{5}{2}, \frac{45}{16}, \frac{50}{16}$.

Ex. 2. Divide $\frac{9}{16}$ by 3, 4, 5. Ans. $\frac{3}{16}, \frac{9}{64}, \frac{9}{80}$.

Art. 21. It follows from the manner of representing fractions, that when the numerator is increased, the value of the fraction becomes greater, but when the denominator is increased the value becomes less. If however the numerator and denominator be both increased or diminished in the same proportion, the value of the fraction is not altered; in other words, if we multiply *both* by any number whatever, or divide them *both* by any number which measures them both, the value of the fraction will not be altered.

Thus $\frac{3}{8} = \frac{6}{16} = \frac{9}{24} = \frac{12}{32}$ &c., multiplying by 2, 3, 4, respectively.

and $\frac{24}{72} = \frac{12}{36} = \frac{8}{24} = \frac{6}{18}$ &c., dividing by 2, 3, 4, respectively.

For if the numerator be multiplied by any number, the fraction is multiplied by that number (Art. 19.); and if the denominator be multiplied by the same number, the fraction is divided by it (Art. 20.); and if any quantity be both multiplied and divided by the same number, its value is not altered. (Vid. Art. 29. Alg.)

And similarly, when the numerator and denominator are both divided by the same number. Hence it follows that every fraction may be expressed in a variety of forms, which have all the same signification.

Art. 22. To express an integer as a fraction having a given denominator.

Multiply the proposed number by the given denominator, and the product will be the numerator of the fraction required.

Ex. Express 9 as a fraction with denominator 6.

$$9 \times 6 = 54;$$

$$\therefore \text{ the fraction is } \frac{54}{6}.$$

The number 9 expressed as a fraction is $\frac{9}{1}$. And since the value of this is not altered by multiplying the numerator and denominator by the same number (Art. 21.)

$$\therefore \frac{9}{1} = \frac{54}{6}.$$

Ex. Express 3, 6, 8, 12, 15, 27, as fractions with denominators 4, 7, 5, 10, 13, 27, respectively.

Ans. $\frac{12}{4}, \frac{42}{7}, \frac{40}{5}, \frac{120}{10}, \frac{195}{13}, \frac{729}{27}$.

Art. 23. To express a mixed number as an improper fraction.

Multiply the integer by the denominator of the fraction, and to the product add the numerator. The sum is the numerator of the improper fraction sought, and is placed above the given denominator.

Ex. Express $5\frac{2}{3}$ as an improper fraction.

$5 \times 3 + 2 = 15 + 2 = 17$ the required numerator, and the required fraction is $\frac{17}{3}$.

For (by last Art.) $5 = \frac{15}{3}$

$$\text{and } \therefore 5\tfrac{2}{3} \text{ or } \frac{15}{3} + \frac{2}{3} = \frac{17}{3}.$$

Ex. 1. Express $16\frac{3}{4}$, $69\frac{2}{5}$ as improper fractions.

Ans. $\frac{67}{4}, \frac{347}{5}$.

Ex. 2. Express $27\frac{2}{9}$, $13\frac{4}{5}$, $514\frac{5}{16}$, as improper fractions.

Ans. $\frac{245}{9}$, $\frac{69}{5}$, $\frac{8229}{16}$.

Ex. 3. Express $333\frac{5}{7}$, $719\frac{11}{12}$, $4128\frac{4}{11}$, as improper fractions.

Ans. $\frac{2336}{7}$, $\frac{8639}{12}$, $\frac{45412}{11}$.

Art. 24. To express an improper fraction as a whole or mixed number.

Divide the numerator by the denominator; the quotient will be the whole number; and the remainder, if any, will be the numerator of the fractional part of the mixed number required, and is placed over the original denominator.

Ex. Express $\frac{36}{4}$, and $\frac{112}{17}$ as a whole number and mixed number respectively. $\frac{36}{4} = 9$. $\frac{112}{17} = 6\frac{10}{17}$.

This Art., which is nothing more than Simple Division, is the converse of the former, and the reason may be illustrated in the same manner.

All improper fractions should be expressed as whole or mixed numbers.

Ex. 1. Express as whole or mixed numbers $\frac{9}{8}$, $\frac{15}{3}$, $\frac{99}{20}$, $\frac{72}{6}$.

Ans. $1\frac{1}{8}$, 5, $4\frac{19}{20}$, 12.

Ex. 2. Express as whole or mixed numbers $\frac{129}{7}$, $\frac{1245}{22}$, $\frac{245}{9}$.

Ans. $18\frac{3}{7}$, $56\frac{13}{22}$, $27\frac{2}{9}$.

Ex. 3. Express as whole or mixed numbers $\frac{76}{4}$, $\frac{111}{5}$, $\frac{5913}{59}$, $\frac{8229}{16}$.

Ans. 19, $22\frac{1}{5}$, $100\frac{13}{59}$, $514\frac{5}{16}$.

ART. 25. To reduce a fraction to lower terms.

Divide both the numerator and denominator by any number which measures them both, and place the quotients as a fraction.

Ex. 1. Reduce $\frac{135}{360}$ to lower terms.

$$\frac{135}{360}=\frac{27}{72}.$$

The numbers 135 and 360 are both divisible by 5, (Vid. Art. 13.) and the fraction $\frac{27}{72}$ thus obtained is still of equal value to $\frac{135}{360}$ by Art. 21.

Ex. 2. Reduce $\frac{30}{125}$, $\frac{24}{16}$, $\frac{128}{324}$ to lower terms.

Ans. $\frac{6}{25}$, $\frac{6}{4}$, $\frac{64}{162}$.

When a fraction is reduced as much as possible by division, so that no common factor except unity remains in the numerator and denominator, the fraction is reduced to its *lowest* terms. In common practice, it is always necessary to reduce fractions to their lowest terms.

ART. 26. To reduce a fraction to its lowest terms.

Divide both the numerator and the denominator by any factor common to both, and let this process be repeated until they have no common factor except unity.

Ex. Reduce $\frac{150}{350}$ to its lowest terms.

Dividing the numerator and denominator by 10 and 5, it $=\frac{15}{35}=\frac{3}{7}$ and this latter fraction is evidently in its lowest terms.

Ex. 1. Reduce $\frac{135}{360}$, $\frac{720}{864}$, $\frac{324}{396}$ to their lowest terms.

Ans. $\frac{3}{8}$, $\frac{5}{6}$, $\frac{9}{11}$.

Ex. 2. Reduce $\frac{24}{32}$, $\frac{30}{125}$, $\frac{208}{684}$ to their lowest terms.

Ans. $\frac{3}{4}$, $\frac{6}{25}$, $\frac{52}{171}$.

Ex. 3. Reduce $\frac{192}{576}$, $\frac{5184}{6912}$, $\frac{1872}{2016}$ to their lowest terms.

Ans. $\frac{1}{3}$, $\frac{3}{4}$, $\frac{13}{14}$.

If however the numerator and denominator be divided by their Greatest Common Measure, the fraction will be immediately reduced to its lowest terms. We shall therefore give the rule for finding this Measure.

ART. 27. To find the Greatest Common Measure (G. C. M.) of two numbers.

A number is said to be a *measure* or *factor* of another, when it divides it exactly. Thus 2 and 4 are measures of 8.

A number is said to be a *common measure* (C. M.) of 2 or more numbers, when it divides them all exactly. Thus 2 is a Common Measure of 4 and 8. And the greatest number which thus divides two or more numbers is called their *greatest common measure.* (G. C. M.) Thus 4 is the greatest common measure of 4 and 8.

To find the greatest common measure of two numbers, divide the greater by the less, and the preceding divisor by the remainder continually, till nothing remains; the last divisor is the greatest common measure.

Ex. 1. Required the greatest common measure of 475 and 589.

```
475) 589 (1
     475
     114) 475 (4
          456
           19) 114 (6
               114
```

Here we divide 589 by 475, and so on continually, and the last divisor 19 is the greatest common measure required.

Obs. If the last divisor be 1, it indicates that the numbers

have no common measure except unity. Also, if the numbers have cyphers in the unit's place as 340, 220; the operation may be abbreviated by finding the greatest common measure of 34 and 22, and then multiplying it by 10; since it is evident that 10 multiplied by the greatest common measure of 34 and 22, will be the greatest common measure of 340 and 220.

To explain the reason of the above rule, we must observe, that any number which measures two others, will also measure their sum and difference, and will measure their sum and difference multiplied by any number. Thus any number which measures 589 and 475 will measure their difference 114, and therefore 114×4 or 456; and any number which measures 475 and 456 will also measure their difference 19. Consequently no number greater than 19 can measure the original numbers 589 and 475.

Again, 19 will measure them both; for it measures 114, and therefore 4×114 or 456, and therefore it will also measure $19 + 456$, or 475; and since it measures 475 and 114 it will also measure their sum 589. Since therefore 19 will measure them both, and no number greater than 19 can measure them both; it follows that 19 will be the greatest common measure. The reason of this rule will be better understood by referring to Art. 51. of the Algebra.

Ex. 1. Find the greatest common measure of 78 and 324.
Ans. 6.

Ex. 2. Find the greatest common measure of 3860 and 4768.
Ans. 4.

Ex. 3. Find the greatest common measure of 2431 and 770.
Ans. 11.

Ex. 4. Find the greatest common measure of 6327 and 23997.
Ans. 57.

Art. 28. To reduce a fraction to its lowest terms by means of the greatest common measure of the numerator and denominator.

Divide the numerator and denominator of the fraction by their greatest common measure, and place the quotients as a fraction.

Ex. Reduce $\frac{475}{589}$ to its lowest terms.

The greatest common measure of 475 and 589 is 19, therefore dividing the numerator and denominator by 19, we have

$$\frac{475}{589} = \frac{25}{31}.$$

Obs. If the numerator and denominator have no common measure except 1, the fraction is already in its lowest terms.

Ex. 1. Reduce $\frac{3860}{4768}$ to its lowest terms. Ans. $\frac{965}{1192}$.

Ex. 2. Reduce $\frac{321}{749}$, $\frac{299}{529}$ to their lowest terms.

Ans. $\frac{3}{7}$, $\frac{13}{23}$.

Ex. 3. Reduce $\frac{1729}{5850}$, $\frac{6409}{7395}$ to their lowest terms.

Ans. $\frac{133}{450}$, $\frac{13}{15}$.

Ex. 4. Reduce $\frac{8398}{29393}$ to its lowest terms. Ans. $\frac{2}{7}$.

Ex. 5. Reduce $\frac{5544}{6552}$ to its lowest terms. Ans. $\frac{11}{13}$.

Ex. 6. Reduce $\frac{11050}{35581}$ to its lowest terms. Ans. $\frac{50}{161}$.

Ex. 7. Reduce $\frac{2382}{2919}$ to its lowest terms. Ans. $\frac{794}{973}$.

Ex. 8. Reduce $\frac{3872}{92807}$ to its lowest terms. Ans. $\frac{32}{767}$.

Art. 29. Of the two methods of reducing a fraction to its lowest terms (Art. 26 and 28), the former is to be generally preferred, since by means of Art. 13., many common measures may be discovered by inspection, and the fractions readily reduced.

The latter method should only be adopted when we are unable to detect by inspection the common measures of the numerator and denominator.

Obs. The greatest common measure of three numbers is found by first finding the greatest common measure of two, and then the greatest common measure of this and the third, which will be the greatest common measure of the three; and similarly if there be more than three numbers.

ART. 30. To reduce fractions of different denominators to others of equal value that have a common denominator.

Multiply the numerator of each fraction by all the denominators except its own; the products are the numerators to the respective fractions sought. Multiply all the denominators into each other; the product is the common denominator.

Reduce $\frac{4}{5}$, $\frac{7}{9}$, $\frac{3}{8}$, to a common denominator.

$4 \times 9 \times 8 = 288$, first numerator,

$7 \times 5 \times 8 = 280$, second numerator,

$3 \times 5 \times 9 = 135$, third numerator,

$5 \times 9 \times 8 = 360$, common denominator;

and the required fractions are $\frac{288}{360}$, $\frac{280}{360}$, $\frac{135}{360}$, or the operation may stand thus:

$$\frac{4}{5} = \frac{4 \times 9 \times 8}{5 \times 9 \times 8} = \frac{288}{360},$$

$$\frac{7}{9} = \frac{7 \times 5 \times 8}{5 \times 9 \times 8} = \frac{280}{360},$$

$$\frac{3}{8} = \frac{3 \times 5 \times 9}{5 \times 9 \times 8} = \frac{135}{360}.$$

The numerators and denominators of each fraction being multiplied by the same number, the value of the fraction will not be altered. (Art. 21.)

Obs. When the numerator of a fraction consist of numbers connected with the sign of addition or subtraction, each of these numbers must be divided by the denominator. Thus,

$$\frac{400 + 70 - 6}{2} = 200 + 35 - 3 = 232.$$

But if the numerator consists of numbers connected with the sign of multiplication, then only one of them must be divided, Thus,

$$\frac{2 \times 7 \times 14}{2} = 7 \times 14 = 98.$$

The reason of this will appear by performing the addition and subtraction in the first Example, and the multiplication in the second, and then dividing by 2.

Ex. 1. Reduce $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$ to a common denominator.

Ans. $\frac{32}{64}$, $\frac{48}{64}$, $\frac{40}{64}$.

Ex. 2. Reduce $\frac{4}{5}$, $\frac{2}{3}$, $\frac{3}{7}$, $\frac{1}{8}$, to a common denominator.

Ans. $\frac{672}{840}$, $\frac{560}{840}$, $\frac{360}{840}$, $\frac{105}{840}$.

Ex. 3. Reduce $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{2}{19}$, to a common denominator.

Ans. $\frac{285}{570}$, $\frac{190}{570}$, $\frac{342}{570}$, $\frac{60}{570}$.

Ex. 4. Reduce $\frac{18}{29}$, $\frac{23}{38}$, to a common denominator.

Ans. $\frac{684}{1102}$, $\frac{667}{1102}$.

The fractions thus obtained may be reduced to lower terms, if the several numerators and denominators have a common measure greater than 1. Sometimes a number of this kind will occur on inspection for a divisor; thus in Ex. 1, all the fractions are divisible by 8, and when reduced to their *least* common denominator, are $\frac{4}{8}$, $\frac{6}{8}$, $\frac{5}{8}$.

And it is evident that when the fractions are reduced to a common denominator, they may be reduced to their *least* common denominator, by dividing all the numerators and the common denominator by their greatest common measure. But the fractions can *at once* be reduced to their least common denominator by finding the Least Common Multiple (L. C. M.) of the denominators.

ART. 31. To find the least common multiple of two or more numbers.

One number is said to contain, or be a *multiple* of another, when it can be divided by it exactly.

Thus 8 is a multiple of 2, 4, and 8.

A common multiple of 2 or more numbers is one which contains each of them; and the *least* such number is called their least common multiple, (L. C. M.) Thus 24 is a common multiple of 2, 3, 4, but 12 is their least common multiple.

To find the least common multiple of 2 or more numbers.

Divide the product of any 2 of the numbers by their greatest common measure, and the quotient will be their least common multiple; proceed in the same way with this quotient and another number; and so on to the last quotient, which will be the required least common multiple.

Obs. If one of the numbers contain another exactly, strike out the number so contained. Thus it is evident that the least common multiple of 24, 16, and 6, will be the least common multiple of 24 and 16, since any number which contains 24 and 16 will also contain 24, 16 and 6.

Find the least common multiple of 6, 12, 16, 18, 24.

Here 24 contains 6 and 12, and therefore the least common multiple of these 5 numbers will be the least common multiple of 16, 18, 24.

Now the least common multiple of 18 and 24 $= \frac{18 \times 24}{6}$ $= 72$, 6 being their greatest common measure.

And the least common multiple of 16 and 72 $= \frac{16 \times 72}{8}$ $= 144$, the required least common multiple.

In order to understand the reason of this Rule we may observe, that if any 2 numbers be multiplied together, their product will be *a* common multiple of each of them; but this product will evidently not be their least common multiple if the numbers have a common measure; in other words, to find

the least common multiple we must divide this product by the greatest common measure of the 2 numbers. (Vide also Art. 52. Algebra.)

Ex. 1. Find the least common multiple of 12, 15, 16.

Ans. 240.

Ex. 2. Find the least common multiple of 30, 75, 93, 153.

Ans. 237150.

Ex. 3. Find the least common multiple of 8, 9, 10, 13.

Ans. 4680.

Ex. 4. Find the least common multiple of 8, 12, 9, 20.

Ans. 360.

ART. 32. To reduce fractions to others of equal value having the least common denominator.

Find the least common multiple of all the denominators, and take this for the common denominator. Multiply the given numerators by the common denominator, and divide the products by the respective given denominators; this will give the corresponding new numerators.

Reduce $\frac{3}{4}$ and $\frac{7}{12}$ to their least common denominator.

The least common multiple of 4 and 12 is 12.

$$\frac{3}{4} = \frac{3 \times 12}{4 \times 12} = \frac{9}{12}, \text{ and the other is } \frac{7}{12}$$

$$\text{or } \frac{3 \times 12}{4} = 9 \text{ new numerator of one,}$$

$$\frac{7 \times 12}{12} = 7 \text{ .... other.}$$

Reduce $\frac{5}{8}$, $\frac{11}{12}$, $\frac{7}{18}$, to their least common denominator.

The least common multiple of 8, 12, and 18, is 72;

$$\therefore \quad \frac{5 \times 72}{8} = 45, \text{ first numerator,}$$

$$\frac{11 \times 72}{12} = 66, \text{ second numerator,}$$

$$\frac{7 \times 72}{18} = 28, \text{ third numerator,}$$

and the fractions are $\frac{45}{72}, \frac{66}{72}, \frac{28}{72}$.

Ex. 1. Reduce $\frac{1}{3}$ and $\frac{4}{27}$ to least common denominator.

Ans. $\dfrac{9}{27}$ and $\dfrac{4}{27}$.

Ex. 2. Reduce $\frac{3}{8}$, $\frac{9}{20}$, $\frac{13}{24}$, to least common denominator.

Ans. $\dfrac{45}{120}$, $\dfrac{54}{120}$, $\dfrac{65}{120}$.

Ex. 3. Reduce $\frac{5}{6}$, $\frac{5}{8}$, $\frac{2}{9}$, $\frac{13}{24}$ to least common denominator.

Ans. $\dfrac{60}{72}$, $\dfrac{45}{72}$, $\dfrac{16}{72}$, $\dfrac{39}{72}$.

Ex. 4. Reduce $\frac{3}{8}$, $\frac{5}{12}$, $\frac{4}{9}$, $\frac{7}{20}$, to least common denominator.

Ans. $\dfrac{135}{360}$, $\dfrac{150}{360}$, $\dfrac{160}{360}$, $\dfrac{126}{360}$.

Ex. 5. Reduce $\frac{11}{19}$, $\frac{18}{29}$, $\frac{23}{38}$, to least common denominator.

Ans. $\dfrac{638}{1102}$, $\dfrac{684}{1102}$, $\dfrac{667}{1102}$.

Ex. 6. Reduce $\frac{1}{828}$, $\frac{1}{1224}$, $\frac{1}{2919}$, to least common denominator.

Ans. $\dfrac{33082}{27391896}$, $\dfrac{22379}{27391896}$, $\dfrac{9384}{27391896}$.

In order to compare the values of fractions, we must always reduce them to a common denominator.

The reduction of fractions to the same denominator is necessary to prepare them for addition or subtraction, but not for multiplication or division.

ART. 33. To reduce a compound fraction to a simple one.

Multiply together all the numerators for a new numerator, and all the denominators for a new denominator.

Ex. 1. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ to a simple fraction.

$2 \times 3 = 6$, new numerator,

$3 \times 5 = 15$, new denominator,

$\therefore$ the simple fraction is $\dfrac{6}{15}$.

For $\dfrac{1}{3}$ of $\dfrac{3}{5} = \dfrac{3}{15}$ (Art. 20.)

$$\therefore \frac{2}{3} \text{ of } \frac{3}{5} = 2 \times \frac{3}{15} = \frac{6}{15} \text{ (Art. 19.)} = \frac{2}{5}.$$

Similarly, mixed numbers must be reduced to improper fractions, and then treated in the same way.

Ex. 2. Reduce $\frac{2}{5}$ of $9\frac{1}{6}$ to a simple fraction.

$$9\tfrac{1}{6} = \frac{55}{6} \text{ (Art. 23.)}$$

$$\therefore \frac{2}{5} \text{ of } 9\tfrac{1}{6} = \frac{2}{5} \text{ of } \frac{55}{6} = \frac{110}{30} = \frac{11}{3} = 3\tfrac{2}{3}.$$

Compound fractions may often be reduced, by striking out factors common to one of the numerators and one of the denominators.

$$\text{Thus } \frac{2}{\cancel{3}} \text{ of } \frac{\cancel{3}}{5} = \frac{2}{5} \text{ as in Ex. 1.}$$

$$\frac{2}{5} \text{ of } 9\tfrac{1}{6} = \frac{\cancel{2}}{\cancel{5}} \text{ of } \frac{\overset{11}{\cancel{55}}}{\underset{3}{\cancel{6}}} = \frac{11}{3} \text{ as in Ex. 2.}$$

Ex. 3. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{8}$ to a simple fraction. Ans. $\frac{1}{4}$.

Ex. 4. Reduce $\frac{3}{4}$ of $\frac{5}{6}$ of $\frac{9}{10}$ to a simple fraction. Ans. $\frac{9}{16}$.

Ex. 5. Reduce $\frac{3}{7}$ of $\frac{91}{135}$ to a simple fraction. Ans. $\frac{13}{45}$.

Ex. 6. Reduce $\frac{13}{17}$ of $\frac{95}{133}$ of $\frac{119}{260}$ to a simple fraction.
Ans. $\frac{1}{4}$.

Ex. 7. Reduce $\frac{2}{5}$ of $3\frac{1}{11}$ of $9\frac{1}{6}$ to a simple fraction.
Ans. $11\frac{1}{3}$.

Ex. 8. Reduce $\frac{3}{4}$ of $\frac{5}{8}$ of $7\frac{1}{9}$ to a simple fraction. Ans. $3\frac{1}{3}$.

In this, as in other examples, fractions must be reduced to their lowest terms, and improper fractions to their equivalent mixed numbers.

Obs. Complex fractions are reduced to simple ones by means of Division. (See Art. 37.)

ADDITION.

ART. 34. Reduce the fractions, if necessary, to their least common denominator; add together the numerators, and place the sum above the common denominator.

Ex. 1. Add together $\frac{3}{5}$ and $\frac{2}{9}$.

$$\frac{3}{5} = \frac{27}{45}, \ \frac{2}{9} = \frac{10}{45};$$

$$\therefore \ \frac{3}{5} + \frac{2}{9} = \frac{27}{45} + \frac{10}{45} = \frac{37}{45}.$$

The reason of this is evident. The fractions $\frac{3}{5}$ and $\frac{2}{9}$ are reduced to their equivalent fractions $\frac{27}{45}$ and $\frac{10}{45}$; and the unit in these latter fractions being divided into 45 equal parts, 27 of those parts together with 10 of them must make 37 such parts.

Obs. If the fractions have a common denominator, then we add the numerators at once. Thus $\frac{2}{5} + \frac{4}{5} = \frac{6}{5} = 1\frac{1}{5}$. It must be remembered that the numerators of fractions that have the same denominator signify like parts, and the reason for adding them is equally obvious, as that for adding integers together.

Mixed numbers may be added by annexing the sum of the *fractions* to the sum of the *integers*. If the sum of the former be a mixed number, its integer is added to the other integers.

Ex. 2. Add together $4\frac{2}{9}$, $6\frac{5}{13}$ and $7\frac{8}{19}$.

The sum of integers $= 4 + 6 + 7 = 17$

.... fractions $= \frac{2}{9} + \frac{5}{13} + \frac{8}{19}$.

$$= \frac{2 \times 13 \times 19}{2223} + \frac{5 \times 9 \times 19}{2223} + \frac{8 \times 9 \times 13}{2223}$$

2223 being the least common multiple of 9, 13, and 19.

$$= \frac{494}{2223} + \frac{855}{2223} + \frac{936}{2223}$$

$$= \frac{2285}{2223} = 1\frac{62}{2223};$$

$$\therefore \text{ required sum} = 17 + 1\frac{62}{2223} = 18\frac{62}{2223}.$$

Similarly, improper fractions should be reduced to mixed numbers, and compound fractions to simple ones, and then added as above.

Ex. 3. Add together $\frac{9}{7}$, and $\frac{2}{3}$ of $\frac{3}{4}$.

$$\frac{9}{7} + \frac{2}{3} \text{ of } \frac{3}{4} = 1\tfrac{2}{7} + \frac{2}{4} = 1 + \frac{2}{7} + \frac{1}{2};$$

$$= 1 + \frac{4}{14} + \frac{7}{14} = 1\tfrac{11}{14}.$$

Ex. 4. Add together $\frac{23}{7}$, $9\frac{5}{11}$, and $2\frac{3}{4}$ of $2\frac{2}{11}$ of $\frac{5}{8}$.

$$\frac{23}{7} + 9\tfrac{5}{11} + 2\tfrac{3}{4} \text{ of } 2\tfrac{2}{11} \text{ of } \frac{5}{8} = 3\tfrac{2}{7} + 9\tfrac{5}{11} + 3\tfrac{3}{4}$$

$$= 15 + \frac{2}{7} + \frac{5}{11} + \frac{3}{4}$$

$$= 15 + \frac{88}{308} + \frac{140}{308} + \frac{231}{308} = 15 + \frac{459}{308} = 16\tfrac{151}{308}.$$

Ex. 5. Add together $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$. Ans. $2\frac{13}{60}$.

Ex. 6. Add together $7\frac{2}{3}$ and $\frac{2}{5}$. Ans. $8\frac{1}{15}$.

Ex. 7. Add together $\frac{5}{7}$, $\frac{8}{9}$ and $1\frac{9}{10}$. Ans. $3\frac{317}{630}$.

Ex. 8. Add together $17\frac{5}{12}$, $\frac{4}{15}$, and $144\frac{11}{21}$. Ans. $162\frac{29}{140}$.

Ex. 9. Add together $\frac{3}{6}$, $2\frac{1}{7}$, and $13\frac{3}{10}$. Ans. $15\frac{33}{35}$.

Ex. 10. Add together $387\frac{1}{2}$, $285\frac{1}{4}$, $394\frac{1}{3}$, and $\frac{2}{5}$ of 3704. Ans. $2548\frac{41}{60}$.

Ex. 11. Add together $\frac{2}{7}$, $4\frac{1}{3}$, and $\frac{3}{5}$ of 2. Ans. $5\frac{86}{105}$.

Ex. 12. Add together $\frac{2}{3}$ of $\frac{7}{11}$, and $\frac{19}{23}$ of $11\frac{1}{4}$. Ans. $9\frac{2179}{3036}$.

Ex. 13. Add together $\frac{7}{15}$, $\frac{10}{21}$ and $\frac{16}{35}$. Ans. $1\frac{14}{35}$.

Ex. 14. Add together $\frac{2}{5}$, $\frac{35}{80}$, $\frac{14}{100}$, $\frac{3}{4}$ of $\frac{1}{35}$, and $\frac{3}{2800}$. Ans. 1.

Ex. 15. Add together $\frac{1}{3}$ of $\frac{5}{7}$, $\frac{1}{4}$ of $\frac{19}{21}$, and $\frac{5}{6}$ of $\frac{9}{14}$. Ans. 1.

Ex. 16. Add together $5\frac{3}{4}$, $\frac{3}{5}$ of $7\frac{1}{2}$, and $8\frac{3}{10}$. Ans. $18\frac{11}{20}$.

SUBTRACTION.

ART. 35. Reduce the fractions, if necessary, to their least common denominator; subtract the numerator of the subtrahend from the numerator of the minuend, and place the remainder above the common denominator.

Ex. 1. Subtract $\frac{2}{7}$ from $\frac{5}{12}$.

$$\frac{2}{7} = \frac{24}{84}, \quad \frac{5}{12} = \frac{35}{84};$$

$$\therefore \frac{5}{12} - \frac{2}{7} = \frac{35}{84} - \frac{24}{84} = \frac{11}{84}.$$

The reason of this operation is similar to that for the addition of fractions. Art. 34.

To subtract a fraction from an integer, subtract the numerator from the denominator, and place the remainder above the denominator. Prefix to this the integer diminished by unity.

Ex. 2. $12 - \frac{3}{5} = 11\frac{2}{5}$, or

$$12 - \frac{3}{5} = \frac{60}{5} - \frac{3}{5} = \frac{57}{5} = 11\frac{2}{5}.$$

To subtract mixed numbers; proceed with the fractions by the foregoing rule, and with the integers in the common method.

Ex. 3. Subtract $2\frac{1}{2}$ from $5\frac{5}{8}$.

$$\text{Here } 5 - 2 = 3,$$

$$\text{and } \frac{5}{8} - \frac{1}{2} = \frac{5}{8} - \frac{4}{8} = \frac{1}{8};$$

and the result is $3\frac{1}{8}$.

Sometimes the numerator of the fraction in the subtrahend exceeds that in the minuend as in

Ex. 4. Subtract $145\frac{7}{9}$ from $238\frac{3}{5}$.

$$\frac{3}{5} - \frac{7}{9} = \frac{27}{45} - \frac{35}{45} = -\frac{8}{45},$$

$$238 - 145 = 93;$$

and the result is $93 - \dfrac{8}{45} = 92\dfrac{37}{45}$.

Similarly, improper fractions should be reduced to mixed numbers, and compound fractions to simple ones, and then subtracted as above.

Obs. These operations will be fully understood, if, when the fractions are reduced to a common denominator, the numerators of the fractions are placed in a column like a lower denomination, and added and subtracted as integers, carrying or borrowing according to the value of the higher denominations.

Ex. 5. Find the value of $\frac{7}{11} - \frac{3}{13}$. Ans. $\frac{58}{143}$.

Ex. 6. Find the value of $33\frac{4}{7} - 21\frac{4}{9}$. Ans. $12\frac{8}{63}$.

Ex. 7. Find the value of $\frac{19}{20} - \frac{1}{7}$ of $\frac{2}{3}$. Ans. $\frac{359}{420}$.

Ex. 8. Find the value of $64\frac{1}{4} - \frac{2}{3}$ of $\frac{3}{4}$. Ans. $63\frac{3}{4}$.

Ex. 9. Find the value of $\frac{1}{2} + \frac{2}{3} - \frac{1}{6} + \frac{3}{8} - \frac{1}{12}$. Ans. $1\frac{7}{24}$.

Ex. 10. Find the value of $\frac{5}{6} - \frac{3}{4} + \frac{2}{3} - \frac{1}{2}$. Ans. $\frac{1}{4}$.

Ex. 11. Find the value of $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{5}{6} - \frac{7}{8} + \frac{11}{12}$.
Ans. $1\frac{7}{24}$.

Ex. 12. Find the value of $\frac{3}{8}$ of $\frac{4}{7} - \frac{2}{11}$ of $3\frac{1}{7} + \frac{5}{9}$ of $3\frac{3}{8}$.
Ans. $1\frac{29}{56}$.

Ex. 13. Find the value of $\frac{5}{3}$ of $2\frac{7}{9} - 3\frac{17}{18}$. Ans. $\frac{37}{54}$.

Ex. 14. Find the value of $5\frac{1}{3}$ of $4\frac{1}{2} - 3\frac{1}{4}$ of $3\frac{1}{5}$. Ans. $13\frac{3}{5}$.

MULTIPLICATION.

Art. 36. Multiply together all the numerators of the fractions for a new numerator; and all the denominators for a new denominator. (Vid. Algebra, Art. 35.)

Ex. 1. Multiply $\frac{2}{3}$ by $\frac{5}{7}$.

$$\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}.$$

This is in fact nothing more than finding the value of the compound fraction $\frac{2}{3}$ of $\frac{5}{7}$, (Art. 33.) And as the multiplication

of a fraction by an integer is effected by taking the fraction a certain number of times, so the multiplication of a fraction by a fraction is effected by taking one fraction so many *parts* of times as the other expresses. So that $\frac{2}{3} \times \frac{5}{7}$ means 2 third-parts of $\frac{5}{7}$, in the same way as $2 \times \frac{5}{7}$ means 2 times $\frac{5}{7}$. Now 1 third part of $\frac{5}{7}$ is $\frac{5}{21}$, therefore 2 third parts of $\frac{5}{7} = \frac{10}{21}$.

The product arising from the multiplication of these fractions may evidently be multiplied by another fraction, and thus the process may be extended to any number of fractions.

Ex. 2. Multiply $\frac{2}{3}$ of $\frac{5}{7}$ by $\frac{6}{11}$.

$$\frac{2}{\cancel{3}} \times \frac{5}{7} \times \frac{\overset{2}{\cancel{6}}}{11} = \frac{20}{77}.$$

Since the method of multiplication of fractions is the same as that for reducing a compound fraction to a simple one, therefore we must as in Art. 33. reduce mixed numbers to improper fractions, and simplify the result by striking out factors common to numerator and denominator.

Ex. 3. Multiply $8\frac{2}{5}$ by $7\frac{3}{4}$.

$$8\tfrac{2}{5} = \frac{42}{5} \qquad 7\tfrac{3}{4} = \frac{31}{4};$$

$$\therefore\ 8\tfrac{2}{5} \times 7\tfrac{3}{4} = \frac{42}{5} \times \frac{31}{4} = \frac{1302}{20} = 65\tfrac{1}{10}.$$

If however we have to multiply a mixed number by an integer, it will be simpler first to multiply the integral part, and then the fractional part, adding the products.

Ex. 4. Multiply $5\frac{1}{4}$ by 138.

$$5 \times 138 = 690,$$

$$\tfrac{1}{4} \times 138 = 34\tfrac{1}{2},$$

$$\therefore\ 5\tfrac{1}{4} \times 138, \text{ or } (5 + \tfrac{1}{4}) \times 138 = 724\tfrac{1}{2}.$$

Ex. 5. Multiply 3756 by $\frac{3}{5}$. Ans. $2253\frac{3}{5}$.

Ex. 6. Multiply $\frac{7}{9}$ by $\frac{2}{3}$. Ans. $\frac{14}{27}$.

Ex. 7. Multiply $\frac{3}{4}$ of $\frac{2}{3}$ by $\frac{2}{3}$ of $\frac{1}{3}$. Ans. $\frac{1}{9}$.

Ex. 8. Multiply $9\frac{1}{2}$ by $\frac{1}{3}$. Ans. $3\frac{1}{6}$.

Ex. 9. Multiply $15\frac{3}{8}$ by $\frac{2}{9}$. Ans. $3\frac{5}{12}$.

Ex. 10. Find the product of $6\frac{3}{10}$, $7\frac{2}{9}$, and $43\frac{7}{13}$. Ans. 1981.

Ex. 11. Find the product of $\frac{3}{8}$, $2\frac{1}{10}$, $1\frac{1}{63}$, $3\frac{5}{87}$.
Ans. $2\frac{35}{87}$.

Ex. 12. Find the product of $\frac{12}{13}$, $2\frac{2}{3}$, $1\frac{1}{25}$, $1\frac{11}{64}$. Ans. 3.

Ex. 13. Find the product of $\frac{428}{515}$, $\frac{5253}{1819}$, and $\frac{5}{4}$. Ans. 3.

Ex. 14. Find the value of $\frac{3}{8}$ of $\frac{4}{7} - \frac{2}{11}$ of $3\frac{1}{7} + \frac{5}{9}$ of $3\frac{3}{8}$.
Ans. $1\frac{29}{56}$.

Ex. 15. Find the value of $\frac{1}{2}$ of $\frac{7}{12}$ of $\frac{3}{5} \times \frac{4}{11}$ of $3\frac{1}{7}$.
Ans. $\frac{1}{5}$.

Ex. 16. Multiply $\frac{7}{8}$ of a penny by 35. Ans. 2*s.* $6\frac{5}{8}$*d.*

Ex. 17. Multiply $\frac{3}{9}$ of a shilling by 84. Ans. £1. 8*s.*

Obs. It is evident that a fraction multiplied by unity will be unaltered; by an integer, increased; by a proper fraction, diminished.

DIVISION.

ART. 37. Invert the divisor and then proceed as in multiplication. (Vid. Algebra, Art. 36.)

Ex. 1. Divide $\frac{2}{5}$ by $\frac{7}{9}$.

$$\frac{2}{5} \div \frac{7}{9} = \frac{2}{5} \times \frac{9}{7} = \frac{18}{35}.$$

To explain the reason of this, let us suppose it required to divide $\frac{2}{5}$ by 7. The value of this will be $\frac{2}{35}$ Art. 20. Now since $\frac{7}{9}$ is 9 times less than 7, the quotient of any fraction divided by $\frac{7}{9}$ will be 9 times greater than the quotient of the same fraction divided by 7. Hence $\frac{2}{5}$ divided by $\frac{7}{9} = \frac{2}{35} \times 9 = \frac{18}{35}$, and this result we see is obtained by inverting the divisor and proceeding as in multiplication.

It is also evident if we have to divide $\frac{2}{5}$ by $\frac{7}{9}$, that the quotient multiplied by $\frac{7}{9}$ will equal $\frac{2}{5}$, as in the division of integers. Hence multiplying each of these equals by $\frac{9}{7}$, we have, quotient equal to $\frac{2}{5} \times \frac{9}{7}$.

If the divisor and dividend have the same denominators, it is sufficient to divide the numerators.

Ex. 2. Divide $\frac{12}{17}$ by $\frac{3}{17}$.

$$\frac{12}{17} \div \frac{3}{17} = \frac{12}{17} \times \frac{17}{3} = 4.$$

Mixed numbers must be reduced to improper fractions, and compound fractions to simple ones, and then divided as above.

Ex. 3. Divide $\frac{18}{217}$ by $1\frac{2}{7}$.

$$1\tfrac{2}{7} = \tfrac{9}{7},$$

$$\frac{18}{217} \div \frac{9}{7} = \frac{\overset{2}{\cancel{18}}}{\underset{31}{\cancel{217}}} \times \frac{\cancel{7}}{\cancel{9}} = \frac{2}{31}.$$

Ex. 4. Divide $5205\frac{1}{5}$ by $\frac{4}{5}$ of 91.

$$5205\tfrac{1}{5} = \frac{26026}{5}, \ \tfrac{4}{5} \text{ of } 91 = \frac{364}{5}.$$

$$\frac{26026}{5} \div \frac{364}{5} = \frac{26026}{364} = 71\tfrac{1}{2}.$$

Similarly, complex fractions may be reduced to simple ones, by division.

Ex. 5. Find the value of $\dfrac{3\frac{1}{13}}{19\frac{2}{43}}$.

$$3\tfrac{1}{13} = \frac{40}{13}, \ 19\tfrac{2}{43} = \frac{819}{43};$$

$$\therefore \ \frac{3\frac{1}{13}}{19\frac{2}{43}} = \frac{40}{13} \div \frac{819}{43} = \frac{40}{13} \times \frac{43}{819} = \frac{1720}{10647}.$$

Ex. 6. Find the value of $\dfrac{2}{3+\dfrac{4}{5+\dfrac{6}{7}}}$.

In this example, the denominator of 2 is $3+\dfrac{4}{5+\dfrac{6}{7}}$.

the denominator of 4 is $5 + \dfrac{6}{7}$.

$$\text{Now } 5 + \frac{6}{7} = \frac{41}{7};$$

$$\therefore \ \frac{4}{5+\frac{6}{7}} = \frac{4}{\frac{41}{7}} = \frac{28}{41};$$

$$\therefore\ 3 + \cfrac{4}{5 + \cfrac{6}{7}} = 3 + \frac{28}{41} = \frac{151}{41};$$

$$\therefore\ \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7}}} = \cfrac{2}{\frac{151}{41}} = \frac{82}{151}.$$

The fraction is sometimes written thus,

$$\frac{2}{3+}\ \frac{4}{5+}\ \frac{6}{7}.$$

$$\text{Similarly } \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{8} = \frac{141}{224}.$$

Fractions of this sort are called *continued fractions*.

Ex. 7. Divide $\frac{14}{27}$ by $\frac{2}{3}$. Ans. $\frac{7}{9}$.
Divide $\frac{9}{16}$ by $4\frac{1}{2}$. Ans. $\frac{1}{8}$.

Ex. 8. Divide $3\frac{1}{6}$ by $9\frac{1}{2}$. Ans. $\frac{1}{3}$.

Ex. 9. Divide 368 by $\frac{5}{7}$. Ans. $515\frac{1}{5}$.

Ex. 10. Divide $\frac{2}{5}$ of $\frac{3}{8}$ by 5. Ans. $\frac{3}{100}$.

Ex. 11. Divide $576\frac{4}{11}$ by 7. Ans. $82\frac{26}{77}$.

Ex. 12. Divide $2\frac{7}{8}$ by $\frac{3}{5}$ of $4\frac{1}{3}$. Ans. $\frac{115}{104}$.

Ex. 13. Find the value of $\frac{64}{75} \times \frac{49}{72} \div \frac{42}{45}$. Ans. $\frac{28}{45}$.

Ex. 14. Find the value of $\frac{2247}{1017} \div \frac{903}{1017} \times \frac{774}{565} \div \frac{1926}{565}$.
Ans. 1.

Ex. 15. Find the value of $\frac{3}{7} \times 1\frac{2}{5} \times 12\frac{1}{2} \div 6\frac{2}{3}$. Ans. $1\frac{1}{8}$.

Ex. 16. Reduce $\dfrac{23\frac{5}{7}}{38}$ to a simple fraction. Ans. $\frac{83}{133}$.

Ex. 17. Reduce $\dfrac{19}{44\frac{1}{3}}$ to a simple fraction. Ans. $\frac{3}{7}$.

Ex. 18. Reduce $\dfrac{4\frac{4}{15} \text{ of } 2\frac{5}{8}}{5\frac{1}{5} - 4\frac{1}{2}}$ to a simple fraction. Ans. 16.

Ex. 19. Divide 5*d*. by $\frac{2}{5}$. Ans. 1*s*. $0\frac{1}{2}$*d*.

Ex. 20. Divide $\frac{2}{5}$ of a shilling by $\frac{4}{5}$. Ans. 6*d*.

REDUCTION.

ART. 38. We may evidently consider shillings and pence as fractions of a pound, and lower denominations of any kind as fractions of higher. And generally in any operation, where different denominations occur, they may be brought to the same denomination by expressing the lower ones as proper fractions of the higher, or the higher as improper fractions of the lower.

ART. 39. To reduce lower denominations to fractions of higher.

Reduce both to the same denomination, and place the given number for the numerator, and the value of the higher for the denominator.

Ex. 1. Reduce 7*d*. to the fraction of a shilling.

$$1 \text{ shilling} = 12d.;$$

$$\therefore \text{ the fraction is } \frac{7}{12}.$$

If reduced to the fraction of £1. it is $\frac{7}{240}$.

Ex. 2. Reduce 15*s*. 7*d*. to the fraction of £1.

$$15s.\ 7d. = 187d., \qquad £1. = 240d;$$

$$\therefore \text{ the fraction is } \frac{187}{240}.$$

$$\text{For } 1d. \text{ is } \frac{1}{240} \text{ of } £1;$$

$$\therefore 15s.\ 7d., \text{ or } 187d. = \frac{187}{240} \text{ of } £1.;$$

Ex. 3. Reduce 27 gallons to the fraction of a barrel of beer.

$$1 \text{ barrel} = 36 \text{ gallons};$$

$$\therefore \text{ the required fraction is } \frac{27}{36} = \frac{3}{4}.$$

Ex. 4. Reduce 1 guinea to the fraction of a moidore.

$$1 \text{ guinea} = 21s.$$

$$1 \text{ moidore} = 27s.;$$

$$\therefore \text{ the required fraction is } \frac{21}{27}.$$

Ex. 5. Reduce 3 cwt. 2 qrs. 3 lbs. to the fraction of a ton.

3 cwt. 2 qrs. 3 lbs. = 395 lbs.

1 ton = 2240 lbs.;

$\therefore$ the required fraction is $\frac{395}{2240} = \frac{79}{448}$.

Ex. 6. Reduce 3*s.* 4*d.* to the fraction of £1. Ans. $\frac{1}{6}$.

Ex. 7. Reduce 7 oz. 4 dwts. to the fraction of a pound Troy. Ans. $\frac{3}{5}$.

Ex. 8. Reduce $\frac{1}{4}$*d.* to the fraction of £1. Ans. $\frac{1}{960}$.

Ex. 9. Reduce 9 oz. 2 dr. to the fraction of a pound Avoirdupois. Ans. $\frac{73}{128}$.

Ex. 10. Reduce 4 lbs. Avoirdupois to the fraction of a cwt. Ans. $\frac{1}{28}$.

Ex. 11. Reduce 6 fur. 16 poles to the fraction of a mile. Ans. $\frac{4}{5}$.

Ex. 12. Reduce 2 roods 20 poles to the fraction of an acre. Ans. $\frac{5}{8}$.

Ex. 13. Reduce 2 weeks 2 days 19 hrs. 12 mins. to the fraction of a month (28 days). Ans. $\frac{3}{5}$.

Ex. 14. Reduce 7 hrs. 12 mins. to the fraction of a day. Ans. $\frac{3}{10}$.

Ex. 15. Reduce 3 yds. 1 ft. to the fraction of 1 mile 4 furlongs. Ans. $\frac{1}{792}$.

Ex. 16. Reduce 10*s.* 5$\frac{3}{4}$*d.* to the fraction of £1. Ans. $\frac{503}{960}$.

Ex. 17. Reduce 5*s.* 6*d.* to the fraction of a guinea. Ans. $\frac{11}{42}$.

Ex. 18. Reduce 9*s.* 10$\frac{1}{2}$*d.* to the fraction of 13*s.* 2$\frac{1}{4}$*d.* Ans. $\frac{158}{211}$.

Ex. 19. Reduce $7\frac{1}{2}d.$ to the fraction of $1s.\ 6d.$ Ans. $\frac{5}{12}$.

Ex. 20. Reduce 12 weeks 3 days 4 hrs. 5 mins. 6 secs. to the fraction of a year, consisting of 365 days 6 hrs.

Ans. $\frac{4184167}{1753200}$.

ART. 40. To reduce higher denominations to fractions of lower.

The rule is the same as in the last Art., but the result in this case will be an improper fraction, and all the examples in Art. 39, will serve as examples for this rule.

Ex. 1. Reduce $1s.$ to the fraction of $7d.$

$$1s. = 12d\ ;$$

$$\therefore \text{ the fraction } \tfrac{12}{7} = 1\tfrac{5}{7}.$$

Ex. 2. Reduce $6s.\ 8\frac{1}{2}d.$ to the fraction of $1\frac{3}{4}d.$ Ans. 46.

Ex. 3. Reduce 14 half-crowns to the fraction of $6s.\ 8d.$

Ans. $5\frac{1}{4}$.

Ex. 4. Reduce $2s.\ 6d.$ to the fraction of £$1s.\ 9d.$ Ans. $1\frac{3}{7}$.

Ex. 5. Reduce £11. $6s.\ 5d.$ to the fraction of £10. $5s.\ 4d.$

Ans. $1\frac{2}{9}$.

ART. 41. To reduce fractions of higher denominations to their proper value.

Multiply the numerator by the value of the given denomination, and divide the product by the denominator; if there be a remainder, multiply it by the value of the next denomination, and continue the division.

Ex. 1. Required the value of $\frac{17}{60}$ of £1.

$$\frac{17}{60} \text{ of £1.} = \frac{17 \times 20}{60} \text{ of 1 sh.} = \frac{17}{3} \text{ sh.} = 5\tfrac{2}{3} \text{ sh.}$$

$$\frac{2}{3} \text{ of 1 sh.} = \frac{2 \times 12}{3} \text{ of } 1d. = 8d.;$$

$$\therefore \frac{17}{60} \text{ of £1.} = 5s.\ 8d.$$

It is evident, that since there are $20s.$ in £1. any fraction of £1. will equal 20 multiplied by the same fraction of a shilling, and similarly for the pence.

Ex. 2. Required the value of $\frac{8}{9}$ of a cwt.

$$\frac{8}{9} \text{ of a cwt.} = \frac{8 \times 4}{9} \text{ of a qr.} = 3\tfrac{5}{9} \text{ qrs.}$$

$$\frac{5}{9} \text{ of a qr.} = \frac{5 \times 28}{9} \text{ of 1 lb.} = 15\tfrac{5}{9} \text{ lbs.}$$

$$\frac{5}{9} \text{ lbs.} = \frac{5 \times 16}{9} \text{ of 1 oz.} = 8\tfrac{8}{9} \text{ ozs.};$$

$$\therefore \frac{8}{9} \text{ of 1 cwt.} = 3 \text{ qrs. } 15 \text{ lbs. } 8\tfrac{8}{9} \text{ ozs.}$$

Ex. 3. Find the value of $\frac{1}{48}$ of a yard.

$$\tfrac{1}{48} \text{ of a yd.} = \frac{1 \times 3}{48} \text{ of a ft.} = \tfrac{1}{16} \text{ of a ft.}$$

$$\tfrac{1}{16} \text{ of a ft.} = \frac{1 \times 12}{16} \text{ of an inch} = \tfrac{3}{4} \text{ inches};$$

$$\therefore \tfrac{1}{48} \text{ of a yd.} = \tfrac{3}{4} \text{ of an inch.}$$

Ex. 4. Find the value of $\frac{3}{4}$, and $\frac{3}{7}$ of £1. Ans. 15*s.* and 8*s.* 6*d.* $3\frac{3}{7}$*f.*

Ex. 5. Find the value of $\frac{2}{5}$ of 1*s.* Ans. 4*d.* $3\frac{1}{5}$*f.*

Ex. 6. Find the value of $\frac{4}{7}$ lb. av. and $\frac{7}{9}$ of a cwt. Ans. 9 oz. $2\frac{2}{7}$ dr., and 3 qrs. 3 lb. 1 oz. $12\frac{4}{9}$ dr.

Ex. 7. Find the value of $\frac{5}{9}$ of an English ell. Ans. 2 qrs. $3\frac{1}{9}$ nls.

Ex. 8. Find the value of $\frac{4}{5}$ of a mile. Ans. 6 fur. 16 pls.

Ex. 9. Find the value of $\frac{5}{8}$ of an acre. Ans. 2 roods 20 pls.

Ex. 10. Find the value of $\frac{3}{5}$ of a month. Ans. 2 wks. 2 days 19 hrs. 12 mins.

Ex. 11. Find the value of $\frac{2}{3}$ of a crown. Ans. 3*s.* 4*d.*

Ex. 12. Find the value of $\frac{7}{18}$ of 1 guinea. Ans. 8*s.* 2*d.*

Ex. 13. Find the value of $\frac{3}{7}$ of £5. 18*s.* 5*d.* Ans. £2. 10*s.* 9*d.*

Ex. 14. Find the value of $\frac{3}{14}$ of half-a-guinea. Ans. 2*s.* 3*d.*

Ex. 15. Find the value of $\frac{2}{3}$ of $\frac{5}{7}$ of 13*s.* 4*d.* Ans. 6*s.* $4\frac{4}{21}$*d.*

Ex. 16. Find the value of $\frac{5}{32}$ of $\frac{16}{3}$ of £273. 2*s.* 6*d.* Ans. £227. 12*s.* 1*d.*

Ex. 17. Find the value of $3\frac{1}{2}$ of $24\frac{1}{8}$ of 6*s.* $0\frac{1}{2}$*d.* Ans. £25. 10*s.* 1*d.* $2\frac{7}{8}$*f.*

ART. 42. To reduce fractions of one denomination to fractions of another.

Express the first denomination (Art. 39, 40.) as a fraction of the other; and then reduce the compound fraction to a simple one.

Ex. 1. Reduce $\frac{7}{8}$ of 1*d.* to the fraction of £1.

$$1d. = \frac{1}{240} \text{ of } £1.;$$

$$\therefore \tfrac{7}{8} \text{ of } 1d. = \tfrac{7}{8} \text{ of } \frac{1}{240} \text{ of } £1.$$

$$= \frac{7}{1920}.$$

Ex. 2. Reduce $\frac{3}{8}$ of 1*s.* to the fraction of a guinea.

$$1s. = \frac{1}{21} \text{ of a guinea};$$

$$\therefore \tfrac{3}{8} \text{ of } 1s. = \frac{3}{8} \times \frac{1}{21} = \frac{1}{56}.$$

Ex. 3. Reduce $2\frac{4}{9}$ of 4 cwt. to the fraction of 3 qrs. 4 lbs.

$$3 \text{ qrs. } 4 \text{ lbs.} = 88 \text{ lbs.}$$

$$4 \text{ cwt.} = 448 \text{ lbs.}$$

$$4 \text{ cwt.} = \frac{448}{88} \text{ of } 3 \text{ qrs. } 4 \text{ lbs.};$$

$$\therefore \text{ the required fraction is } 2\tfrac{4}{9} \times \frac{448}{88} = 12\tfrac{4}{9}.$$

Ex. 4. Reduce $\frac{1}{1289}$ of £1. to the fraction of a shilling, a penny, and a farthing. Ans. $\frac{20}{1289}$, $\frac{240}{1289}$, $\frac{960}{1289}$.

Ex. 5. Reduce $\frac{4}{7}$ of a lb. av. to the fraction of a cwt. Ans. $\frac{1}{196}$.

Ex. 6. Reduce $\frac{4}{5}$ of a dwt. to the fraction of a lb. Troy. Ans. $\frac{1}{300}$.

Ex. 7. Reduce $\frac{1}{196}$ of a cwt. to the fraction of a lb. Ans. $\frac{4}{7}$.

Ex. 8. Reduce $4\frac{3}{4}d.$ $\frac{1}{5}$ to the fraction of 1*s.* Ans. $\frac{2}{5}$.

Ex. 9. Reduce 9 ozs. $2\frac{2}{7}$ drs. to a fraction of 1 lb. Ans. $\frac{4}{7}$.

Ex. 10. Reduce 2 qrs. $3\frac{1}{9}$ nls. to a fraction of an English ell. Ans. $\frac{5}{9}$.

Ex. 11. Reduce $\frac{1}{24}$ of a yard to the fraction of an inch. Ans. $1\frac{1}{2}$.

Ex. 12. Reduce $\frac{3}{8}$ of £1. to the fraction of 1 guinea.

Ans. $\frac{5}{14}$.

Ex. 13. Reduce $\frac{4}{5}$ of a groat to the fraction of 2s. 6d.

Ans. $\frac{2}{25}$.

Ex. 14. Reduce $\frac{2}{5}$ of 2s. $4\frac{1}{2}d.$ to the fraction of 2s. 6d.

Ans. $\frac{57}{150}$.

Ex. 15. Reduce $\frac{7}{13}$ of half-a-crown to the fraction of half-a-guinea.

Ans. $\frac{5}{39}$.

Ex. 16. Reduce $\frac{1}{5}$ of $\frac{6}{11}$ of half-a-crown to the fraction of a guinea.

Ans. $\frac{1}{77}$.

Ex. 17. Reduce $\frac{3}{8}$ sq. ft. to the fraction of a pole.

Ans. $\frac{1}{726}$.

Ex. 18. Reduce $3\frac{1}{2}$ of 2 ac. 3 r. to the fraction of 2 r. $2\frac{1}{2}$ pl.

Ans. $18\frac{2}{3}$.

Ex. 19. Reduce $33\frac{1}{4}$ of 3 qrs. to the fraction of $3\frac{3}{4}$ tons.

Ans. $\frac{133}{400}$.

Ex. 20. Reduce 3s. $8\frac{3}{4}d.$ to the fraction of a guinea.

Ans. $\frac{179}{1008}$.

Ex. 21. Reduce $\frac{3}{4}$ of $\frac{1}{2}$ a guinea to the fraction of £1.

Ans. $\frac{63}{160}$.

Ex. 22. Reduce $\frac{162}{35}$ of 15s. to the fraction of a moidore.

Ans. $2\frac{4}{7}$.

Ex. 23. Reduce $\frac{1}{720}$ of £1. to the fraction of a farthing.

Ans. $1\frac{1}{3}$.

Ex. 24. Reduce $\frac{3}{392}$ of a cwt. to the fraction of a lb.

Ans. $\frac{6}{7}$.

ART. 43. Since by means of Reduction, we are able to express a fraction of a higher denomination in its proper integral value, we may evidently add, subtract, &c. fractions of different denominations.

Ex. 1. Add $\frac{3}{4}$ of £1. to $\frac{5}{6}$ of 1 shilling;

$$\tfrac{3}{4} \text{ of £1.} = \frac{3 \times 20}{4} s. = 15s. \text{ (Art. 14.)}$$

$$\tfrac{5}{6} \text{ of } 1s. = \frac{5 \times 12}{6} d. = 10d.;$$

$\therefore$ required sum is 15s. 10d.

Ex. 2. Subtract $\frac{5}{6}$ of a lb. from $\frac{4}{5}$ of a ton.

$$\tfrac{4}{5} \text{ of a ton} = \frac{4 \times 20}{5} \text{ cwt.} = 16 \text{ cwt.};$$

$$\tfrac{5}{6} \text{ of a lb.} = \frac{5 \times 16}{6} \text{ ozs.} = 13\tfrac{1}{3} \text{ ozs.};$$

$$\tfrac{1}{3} \text{ of an oz.} = \frac{1 \times 16}{3} \text{ drs.} = 5\tfrac{1}{3} \text{ drs.};$$

therefore the difference = 16 cwt. − 13 ozs. − $5\frac{1}{3}$ drs.
= 15 cwt. 3 qrs. 27 lb. 2 ozs. $10\frac{2}{3}$ drs.

Ex. 3. Add $\frac{1}{2}$ of 1*d.* to $\frac{2}{3}$ of £1. Ans. 13*s.* $4\frac{1}{2}$*d.*

Ex. 4. Add $\frac{3}{4}$ of a lb. troy, to $\frac{1}{6}$ of an oz.
Ans. 9 ozs. 3 dwt. 8 gr.

Ex. 5. Add $\frac{1}{6}$ of a yard to $\frac{2}{3}$ of an inch. Ans. 6 in. 8 parts.

Ex. 6. Subtract $\frac{3}{4}$ of 1*s.* from $\frac{3}{4}$ of £1. Ans. 14*s.* 3*d.*

Ex. 7. Subtract $\frac{3}{4}$ of 2*s.* 6*d.* from $\frac{8}{9}$ of a guinea. Ans. 16*s.* $9\frac{1}{2}$*d.*

Ex. 8. Required the sum and difference of $\frac{2}{3}$ of 5 guineas, and $\frac{3}{4}$ of $\frac{7}{9}$ of £1. Ans. £4. 1*s.* 8*d.*; and £2. 18*s.* 4*d.*

Ex. 9. Required the sum and difference of $\frac{1}{3}$ of £1., and $\frac{2}{9}$ of a guinea. Ans. 11*s.* 4*d.*; and 2*s.*

Ex. 10. Reduce $\dfrac{3\frac{1}{6}}{1\frac{1}{13}}\left\{\dfrac{19}{120} \text{ of £1.} - \dfrac{7}{48} \text{ of } 1s.\right\}$ to the fraction of a moidore. Ans. $\dfrac{1885}{5832}$.

Ex. 11. Compare the values of $\frac{1}{19}$ of £1., $\frac{1}{20}$ of a guinea, and $\frac{8}{35}$ of a crown. (Art. 32.) Ans. $\dfrac{2800}{2660}, \dfrac{2793}{2660}, \dfrac{3040}{2660}$.

Ex. 12. Compare the values of $\frac{1}{21}$ of £1., $\frac{1}{22}$ of a guinea, and $\frac{1}{4}$ of 3*s.* $9\frac{1}{2}$*d.* Ans. $\dfrac{7040}{7392}, \dfrac{7056}{7392}, \dfrac{7007}{7392}$.

Ex. 13. Find the value of $\frac{3}{4}$ of a guinea + $\frac{3}{8}$ of a crown, + $\frac{1}{3}$ of 7*s.* 6*d.* − $\frac{3}{4}$ of 2*d.* Ans. £1. 2*s.*

Ex. 14. Reduce $\frac{3}{8}$ of a crown + $\frac{4}{5}$ of a shilling to the fraction of a guinea. Ans. $\frac{107}{840}$.

Ex. 15. Add together $\frac{4}{7}$ cwt., $8\frac{1}{6}$ lbs. $5\frac{9}{10}$ ozs.
Ans. 2 qrs. 17 lbs. $3\frac{7}{30}$ ozs.

Ex. 16. What is the difference between $\frac{1}{12}$ of £1., and $\frac{1}{14}$ of a guinea? Ans. 2*d*.

Ex. 17. What is the difference between $\frac{7}{9}$ of £1., and $\frac{2}{5}$ of a guinea, expressed as a fraction of half a guinea? Ans. $\frac{92}{135}$.

Ex. 18. Divide $18\frac{1}{3}$*s*. by 2*s*. $3\frac{1}{2}$*d*. Ans. 8.

Ex. 19. Divide £69. 17*s*. $5\frac{3}{4}$*d*. by 9. Ans. £7. 15*s*. $3\frac{11}{36}$*d*.

Ex. 20. Multiply £4. 0*s*. $5\frac{2}{3}$*d*. by 9.; and £5. 3*s*. $4\frac{12}{13}$*d*. by 31. Ans. £36. 4*s*. 3*d*.; and £160. 5*s*. $8\frac{8}{13}$*d*.

Ex. 21. Divide £29. 7*s*. $0\frac{1}{2}$*d*. by 7. Ans. £4. 3*s*. $10\frac{5}{14}$.

Ex. 22. Divide £7. 13*s*. 4*d*. by $\frac{12}{77}$. Ans. £49. 3*s*. $10\frac{2}{3}$*d*.

Ex. 23. Divide £99. 17*s*. $0\frac{1}{2}$*d*. by $19\frac{3}{4}$. Ans. £5. 1*s*. $1\frac{1}{4}$*d*. $\frac{45}{79}$.

Ex. 24. A rouble is worth $1\frac{51}{60}$ florin, and a pound is worth $10\frac{3}{4}$ florins. What is the value of a rouble? Ans. 3*s*. $5\frac{13}{43}$*d*.

RULE OF THREE.

Art. 44. Since the relative magnitudes of quantities which constitute any proportion are evidently independent of the manner in which they are expressed; the terms of the proportion may be vulgar fractions as well as integral quantities. And this being the case, the same rules which enabled us to solve questions in the proportion of integers, will enable us to solve questions in the proportion of fractions..

Ex. 1. If $\frac{3}{4}$ of a yard cost $\frac{5}{8}$ of a pound, what will $\frac{9}{10}$ of a yard cost?

$\frac{3}{4}$ of a yard cost $\frac{5}{8}$ of £1.;

$\therefore$ $\frac{3}{2}$ .... $\frac{5}{4}$ of £1., multiplying by 2;

$\therefore$ $\frac{3}{10}$ .... $\frac{1}{4}$ of £1., dividing by 5;

$\therefore$ $\frac{9}{10}$ .... $\frac{3}{4}$ of £1., multiplying by 3;

or $\frac{9}{10}$ of yard cost 15*s*.

Ex. 2. If 48 men can build a wall in $24\frac{1}{4}$ days, how many can do the same in 192 days?

This is an example of inverse proportion.

48 men build the wall in $24\frac{1}{4}$ or $\frac{97}{4}$ days;

$\therefore$ 12 .... 97 days, (dividing and multiplying by 4;)

$\therefore$ 97×12 men build the wall in 1 day, (multiplying and dividing by 97;)

$\therefore$ $\dfrac{97 \times 12}{192}$ men build the wall in 192 days;

or $6\frac{1}{16}$ men build it in 192 days.

Ex. 3. If 9 men spend £$10\frac{7}{9}$. in 18 days, how much will 20 spend in 30 days.

9 men in 18 days spend £$10\frac{7}{9}$., or $\frac{97}{9}$ of £1.;

$\therefore$ 1 man .... $\frac{97}{81}$ of £1.;

$\therefore$ 1 man in 1 .... $\dfrac{97}{81 \times 18}$ of £1.;

$\therefore$ 20 men in 1 .... $\dfrac{20 \times 97}{81 \times 18}$ of £1.;

$\therefore$ 20 men in 30 .... $\dfrac{30 \times 20 \times 97}{81 \times 18}$ of £1.;

or $\dfrac{5 \times 20 \times 97}{81 \times 3}$ of £1.;

or $\dfrac{9700}{243}$ of £1.

$\therefore$ 20 men in 30 days spend £39. 18*s*. $4\frac{20}{81}$*d*.

This is an example of the double Rule of Three in fractions.

Ex. 4. Two pipes fill a cistern in 3 and 4 hours respectively, in what time will they fill it together?

One pipe fills it in 3 hours;
$\therefore$ one pipe fills $\frac{1}{3}$ of it in 1 hour.
The other pipe fills it in 4 hours;
$\therefore$ it fills $\frac{1}{4}$ of it in 1 hour;

$\therefore$ the 2 pipes fill $(\frac{1}{3} + \frac{1}{4})$ of it in 1 hour;

or . . . . $\frac{7}{12}$ . . . .

$\therefore$ the 2 pipes will fill it in $\frac{12}{7}$ hours, or $1\frac{5}{7}$ of an hour.

Ex. 5. *A*, *B*, and *C* do a piece of work separately in 3, 4, 5 days; in what time will they perform it jointly?

A does the work in 3 days;

$\therefore$ *A* does $\frac{1}{3}$ of the work in 1 day;

Similarly, *B* . . . . $\frac{1}{4}$ . . . .

C . . . . $\frac{1}{5}$ . . . .

$\therefore$ *A*, *B*, and *C* together do $\frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, or $\frac{47}{60}$ in 1 day;

$\therefore$ . . . . do the work in $\frac{60}{47}$ days;

or they jointly do it in $1\frac{13}{47}$ days.

Ex. 6. If $\frac{7}{8}$ lb. cost $\frac{3}{4}s.$, how many lbs. will $\frac{8}{9}s.$ buy?

Ans. $1\frac{1}{27}$ lb.

Ex. 7. If $\frac{1}{4}$ of a yard cost $\frac{2}{3}$ of a pound, what will $\frac{3}{5}$ of an English ell cost? Ans. £2.

Ex. 8. The English shilling weighs 3 dwts. 15 grs., of which 3 parts out of 40 are alloy, and the rest pure silver; what weight of pure silver is there in 20 shillings?

Ans. 3 ozs. $7\frac{1}{16}$ dwts.

Ex. 9. If 3 men can do a piece of work in $4\frac{1}{2}$ hours, in how many hours will 10 men do it? Ans. $1\frac{7}{20}$ hour.

Ex. 10. If £100. in twelve months gain £6. interest, what sum will gain £$3\frac{3}{8}$. in 9 months? Ans. £75.

Ex. 11. If the carriage of 60 cwt. 20 miles cost £$14\frac{1}{2}$., what weight can be carried 30 miles for £$5\frac{7}{16}$.? Ans. 15 cwt.

Ex. 12. If $\frac{3}{16}$ of a lottery ticket be worth £4. 10*s.*, what is the value of $\frac{1}{5}$th of it? Ans. £4. 16*s.*

Ex. 13. If $4\frac{5}{9}$ ozs. of tea cost $8\frac{2}{5}s.$, what will $30\frac{3}{7}$ lbs. cost?

Ans. £44. 17*s.* $8\frac{1}{2}d.$ $\frac{86}{205}$.

Ex. 14. A cistern is filled by a tap *A* in $\frac{1}{4}$ hr., and by a tap *B* in 25 minutes; and is emptied by a tap *C* in 30 minutes; in what time will it be filled if the three taps be all open together? Ans. $13\frac{7}{11}$ minutes.

Ex. 15. *A* was owner of $\frac{4}{17}$ of a vessel, and sold $\frac{3}{11}$ of $\frac{2}{9}$ of his share for £$\frac{100}{33}$.; what was the value of $\dfrac{1\frac{2}{8}}{4\frac{1}{4}}$ of $\frac{2}{5}$ of it?

Ans. £100.

Ex. 16. If a person's estate be worth £1384. 16*s.* a year, and the land-tax be assessed at 2*s.* $9\frac{1}{2}$*d.* per pound, what is his annual income?

Ans. £1191. 10*s.* $1\frac{1}{5}$*d.*

CHAPTER III.

DECIMAL FRACTIONS.

Art. 45. In the common system of notation of integers, the actual value of any figure depends upon its position with respect to the place of units; the *local* value of any figure increasing tenfold as we proceed from right to left. Now if on this same system of notation we continue the figures to the *right* of the unit's place, they will in the same manner decrease from left to right, each figure being one-tenth of what it would be, if it stood one place further to the left.

These figures to the right of the unit's place, since they represent fractions having ten or some product of tens for their denominators, are called *decimal* fractions, or *decimals*; and are separated from the integers by a point or dot called the *decimal point*, placed after the unit's place. Thus 12.154 is equivalent to $10 + 2 + \frac{1}{10} + \frac{5}{100} + \frac{4}{1000}$, or $12\frac{154}{1000}$. The first figure after the decimal point signifies tenth parts, the second 100th parts, and so on; and the actual value of any figure may be readily obtained by dividing it by 10, 100, 1000, &c., according as it stands in the first, second, or third place to the right of the units; in other words, by dividing it by 1 prefixed to *one* cypher, if in the first, by 1 prefixed to *two* cyphers, if in the second place, &c.; there always being as many cyphers in the denominator as the figure is *places* to the right of the unit's place.

Art. 46. A *decimal* fraction therefore is a vulgar fraction whose denominator is *ten* or some product of tens; but since its denominator always consists of 1 prefixed to as many cyphers as the numerator has places, this denominator is never set down. Thus, $12.15 = 12\frac{15}{100}$.

The great advantage that decimal fractions have over vulgar fractions is this, that the *latter* cannot easily be compared, because the numbers are divided into different parts; whereas in the *former* the units are divided into like parts, and the divisions and subdivisions are regulated by the same scale that is used in the Arithmetic of integers.

As cyphers when placed to the left hand of *integers* have no signification, but when to the right hand increase their value 10 times each; so cyphers when placed to the right hand of a *decimal* have no signification, but to the left hand diminish their value 10 times each. Thus, $\cdot 5$ and $\cdot 50$ are the same, because they are respectively $= \frac{5}{10}$, and $\frac{50}{100}$, or $\frac{5}{10}$; but $\cdot 05 = \frac{5}{100}$, and $\cdot 005 = \frac{5}{1000}$.

From what has been said, it will be seen that

$$4.7 = 4\tfrac{7}{10};$$

$$\cdot 47 = \frac{4}{10} + \frac{7}{100} = \frac{47}{100};$$

$$\cdot 047 = \frac{4}{100} + \frac{7}{1000} = \frac{47}{1000};$$

$$\cdot 407 = \frac{4}{10} + \frac{7}{1000} = \frac{407}{1000};$$

$$4.07 = 4\frac{7}{100};$$

$$4.007 = 4\frac{7}{1000};$$

in all these cases the denominator having as many cyphers as there are figures to the right of the decimal point, these figures being the numerators.

Art. 47. Any vulgar fraction therefore whose denominator is 10, or some product of tens, may be expressed as a decimal fraction by marking off by a point from the right of the numerator as many figures as there are cyphers in the denominator, prefixing cyphers to the numerator if necessary.

$$\text{Thus, } 4\frac{7}{10} = 4.7;$$

$$\frac{47}{100} = \cdot 47;$$

$$\frac{47}{1000} = \cdot 047, \quad \frac{407}{1000} = \cdot 407;$$

$$4\frac{7}{1000} = 4.007, \quad 4\frac{7}{100} = 4.07;$$

in all these cases there being as many decimal places as there are cyphers in the denominators. Similarly,

$$\frac{12}{100} = \cdot 12, \ \frac{119}{10000} = \cdot 0119, \ \frac{79}{100000} = \cdot 00079.$$

Also from the manner of representing decimals, it is evident that any decimal is multiplied by 10, 100, &c., by moving the point 1, 2, &c., places to the *right*, and divided, by moving it 1, 2, &c., places to the *left*.

Thus, $3.16 \times 10 = 31.6$, since $3\frac{16}{100} \times 10 = 30\frac{16}{10} = 31.6$;

$3.16 \times 100 = 316$, since $3\frac{16}{100} \times 100 = 316$.

Also $475.2 \div 10 = 47.52$,

$475.2 \div 100 = 4.752$;

$$\text{for } 475.2 \div 10 = 475\tfrac{2}{10} \div 10$$

$$= \frac{475}{10} + \frac{2}{100}$$

$$= \frac{4750}{100} + \frac{2}{100} = 47.52;$$

$$\text{and } 475.2 \div 100 = 475\tfrac{2}{10} \div 100$$

$$= \frac{475}{100} + \frac{2}{1000};$$

$$= \frac{4752}{1000} = 4.752.$$

ADDITION.

ART. 48. Arrange the figures according to their local values, placing the units under units, tens under tens, &c., &c., tenth parts under tenth parts, &c., &c.; and add them as in common integers, placing the decimal point in the sum in the same line with the other points.

Ex. 1. Add together 32.035, 116.374, 160.63, 12·3645.

32.035		32.0350
116.374		116.3740
160.63	or	160.6300
12.3645		12.3645
321.4035		321·4035

The value of decimal places decreases like that of integers, ten of the lower place in either being equal to one of the higher; and the same holds in passing from decimals to integers; therefore all operations are performed in the same way with decimals (whether placed by themselves or annexed to integers,) as with common integers; the only peculiarity being in the pointing of the decimals.

Obs. In order to avoid error it is sometimes more simple to fill up the blank places with cyphers, (as above) which being placed to the *right* hand, do not alter their value.

Ex. 2. Add together 72.5 + 32.071 + 2.1574 + 371.4 + 2.75. Ans. 480.8784.

Ex. 3. Add together 3.5 + 47.25 + 927.01 + 1.5. Ans. 979.26.

Ex. 4. Add together ·372 + ·4567 + 14·3713 + 371.01. Ans. 386·21.

Ex. 5. Add together 321.4 + 12 + 31.6154 + ·01 + 2.214 + 415.62. Ans. 782.8594.

Ex. 6. Add together 232·15 + ·721 + 36·999 + 730.45797 + ·00203. Ans. 1000·33.

All these examples may be verified by expressing them as vulgar fractions with a common denominator. Thus,

Ex. 2.
$$72.5 = 72\tfrac{5}{10} = 72\tfrac{5000}{10000};$$
$$32.071 = 32\tfrac{71}{1000} = 32\tfrac{710}{10000};$$
$$2.1574 = 2\tfrac{1574}{10000} = 2\tfrac{1574}{10000};$$
$$371.4 = 371\tfrac{4}{10} = 371\tfrac{4000}{10000};$$
$$2.75 = 2\tfrac{75}{100} = 2\tfrac{7500}{10000};$$
$$\text{and their sum} = 479 + \frac{5000 + 710 + 1574 + 4000 + 7500}{10000}$$
$$= 479\tfrac{18784}{10000} = 479 + 1.8784 = 480.8784.$$

SUBTRACTION.

ART. 49. Arrange the figures according to their local values, placing the units under units, tens under tens, &c., &c., and subtract them as in common integers, placing the decimal point in the difference in the same line with the other two points.

Ex. 1. Subtract 9.2993 from 13.348.

13.348		13.3480
9.2993	or	9.2993
4·0487		4.0487

The reason of this is similar to that for Addition.

The result may be verified thus,

$$13.348 = 13\tfrac{348}{1000} = 13\tfrac{3480}{10000};$$
$$9.2993 = 9\tfrac{2993}{10000};$$
$$\therefore \text{ their difference} = 4\tfrac{487}{10000} = 4.0487.$$

Ex. 2. Subtract 10.6752 from 12.248. Ans. 1.5728.

Ex. 3. Subtract 4.41967 from 37.001. Ans. 32.58133.

Ex. 4. Subtract ·017923 from ·90124. Ans. ·883317.

Ex. 5. Subtract ·000987 from ·0001. Ans. ·0000013.

MULTIPLICATION.

Art. 50. Multiply as in integers, and point off in the product as many decimal places as there are in both factors; prefixing cyphers, if necessary, to the left hand.

Ex. 1. Multiply 1.37 by 1.8.

$$\begin{array}{r} 1.37 \\ 1.8 \\ \hline 1096 \\ 137 \\ \hline 2.466 \end{array}$$

The reason of this rule may be explained by observing, that the value of the product depends upon the value of the factors; and since each decimal place in either factor diminishes its value 10 times, it must equally diminish the value of the product. See however, Algebra, Art. 53.

The result may be verified thus,

$$1.37 = 1\tfrac{37}{100}, \qquad 1.8 = 1\tfrac{8}{10};$$

$$\therefore\ 1.37 \times 1.8 = \frac{137}{100} \times \frac{18}{10} = \frac{2466}{1000} = 2.466.$$

Ex. 2. Multiply 43.75 by .48. Ans. 21.

Ex. 3. Multiply ·1572 by ·12. Ans. ·018864.

Ex. 4. Multiply ·2365 by ·2435. Ans. ·05758775.

Ex. 5. Multiply 3706·205 by 34·005; and 57.296 by 120.
Ans. 126029·501025; and 6875.52.

Ex. 6. Multiply 2.83 by ·013. Ans. ·03679.

Ex. 7. Multiply 36.2 by 4.57. Ans. 165.434.

Ex. 8. Multiply 60 by ·00048; and prove the truth of the result by vulgar fractions. Ans. ·0288.

Ex. 9. Find the value of 2.7 × .27 × ·027 × 270.
Ans. 5.31441.

Ex. 10. The true length of a year being 365.24224 days, find what the error amounts to by the common reckoning in four centuries. Ans. ·104 days.

DIVISION.

ART. 51. Divide as in integers, and point off in the quotient as many decimal places as the number of decimal places in the dividend exceeds the number in the divisor.

If the quotient does not contain a sufficient number of figures to enable us to point off all the decimal places, we must prefix cyphers to the *left* hand.

If the dividend has less decimal places than the divisor, we must annex as many cyphers to the *right* hand of the quotient as make up the difference, and the quotient will be a whole number.

If the number of decimal places in the dividend equal the number in the divisor, the quotient will be a whole number.

If the division leave a remainder, the quotient may be extended to more decimal places; but these will not affect the position of the decimal point.

Ex. 1. Divide 165.434 by 36.2.

```
36.2)165.434(4·57
      2063
       2534
```

Here there are two more decimal places in the dividend than in the divisor, and therefore we point off 2 in the quotient 457, making it 4·57.

Ex. 2. Divide ·03679 by 2.83.

```
2.83)·03679(·013
       849
```

Here there are 3 more decimal places in the dividend than in the divisor, and we therefore prefix one cypher to the left hand of the quotient 13, so that it becomes ·013.

Ex. 3. Divide ·805 by ·0023.

```
·0023)·805(350
       115
```

Here the dividend has one less decimal place than the divisor, and we therefore annex one cypher to the right hand of the quotient 35, so that it becomes 350, a whole number.

Ex. 4. Divide 2617609·5 by 237964·5.

$$237964{\cdot}5\overline{)\,2617609{\cdot}5\,}(11$$
$$\underline{2379645}$$

Here there are the same number of decimal places in the dividend and divisor, and therefore the quotient 11 is a whole number.

Ex. 5. Divide 37.24 by 2.9.

```
2.9) 37.2400 (12.841 &c. &c.
     82
     244
      120
        40
        11 &c.
```

In this Example, after dividing 37.24 by 2.9 the quotient is 128, and since there is one more decimal place in the dividend than divisor, it will be pointed thus, 12.8. But the division does not terminate here, there being a remainder 12; and we therefore extend the quotient to more decimal places by annexing cyphers to the dividend (which does not alter its value, Art. 46), the position of the decimal point remaining the same. It will amount to the same thing if we fix the decimal point *after* annexing cyphers to the dividend. Thus, as far as we have gone in the above Example, there are three more decimal places in the dividend than the divisor, and therefore the quotient 12841 is pointed thus, 12.841.

The reason of the pointing is this. By the nature of division, the product of the divisor and quotient equals the dividend; and therefore we must so point the quotient that the number of decimal places in the dividend may equal the number in the divisor and quotient together; agreeably to last Article. See however Algebra Art. 54.

The result of these Examples may be verified by vulgar fractions; thus, Example 3.

$$\cdot 805 = \frac{805}{1000}, \qquad \cdot 0023 = \frac{23}{10000};$$

$$\therefore \frac{\cdot 805}{\cdot 0023} = \frac{805}{1000} \times \frac{10000}{23}$$

$$= \frac{805 \times 10}{23} = \frac{8050}{23} = 350.$$

Ex. 6. Divide 15 by 6.25.

$$6{\cdot}25)\,\underline{15.000}\,(2{\cdot}4$$
$$\underline{2500}$$

Here we annex cyphers to the dividend before commencing the division, and count them all as decimals.

Ex. 7. Divide 7.59345 by ·345. Ans. 22.01.

Ex. 8. Divide ·007257 by 1.23. Ans. ·0059.

Ex. 9. Divide 16.8 by ·024, and prove the truth of the result by vulgar fractions. Ans. 700.

Ex. 10. Divide 9.065 by ·049. Ans. 185.

Ex. 11. Divide 85643.825 by 6.321. Ans. 13549.0942 &c.

Ex. 12. Divide 721.17562 by 2.257432. Ans. 319.467 &c.

Ex. 13. Divide 60 by ·00048. Ans. 125000.

Ex. 14. Divide 2.86 by ·013. Ans. 220.

Ex. 15. Divide 17.171717 by 343.4. Ans. ·050005.

Ex. 16. Divide 1.540217 by 17. Ans. ·090601.

Ex. 17. Divide 9.6195 by 1.21. Ans. 7.95.

Ex. 18. Divide 6 by ·0025. Ans. 2400.

Ex. 19. Divide 516 by ·0016. Ans. 322500.

Ex. 20. Divide 1 by ·0016. Ans. 625.

Ex. 21. Divide 1 by 3.14159. Ans. ·3183101 &c.

REDUCTION.

ART. 52. To reduce a vulgar fraction to a decimal.

Annex a cypher to the numerator, and divide it by the denominator, annexing a cypher continually to the remainder; the quotient will be the required decimal.

Ex. 1. Reduce $\frac{12}{75}$ to a decimal.

$$\begin{array}{r} 75)\underline{12.0}(\cdot16 \\ \underline{450} \end{array}$$

The reason of the pointing in the quotient is the same as in division. In the above Example the dividend has properly two more cyphers than the divisor, and therefore we mark off two decimal places in the quotient.

Ex. 2. Reduce $\frac{5}{64}$ to a decimal.

$$\begin{array}{r} 64)\underline{5.00}(\cdot078125 \\ \underline{520} \\ \underline{80} \\ \underline{160} \\ \underline{320} \end{array}$$

Here the first figure in the quotient is 0, because we are obliged to annex two cyphers before division; in other words, the first figure in the quotient is 0 with a remainder 50.

Ex. 3. Reduce $\frac{3}{40}$ and $\frac{1}{400}$ to decimals.

$$\frac{3}{40} = \frac{1}{4} \times \frac{3}{10} = \frac{\cdot3}{4} \text{ (Art. 47.)} = \cdot075 \text{ (Art. 51.)}$$

$$\frac{1}{400} = \frac{1}{4} \times \frac{1}{100} = \frac{\cdot01}{4} = \cdot0025.$$

In all cases of this kind where the denominator has ten or some multiple of ten for its denominator, it will be simpler to proceed as in this example.

In these three examples, the divison terminates and the decimals are therefore called *terminate* or *finite* decimals.

Ex. 4. Reduce $\frac{2}{3}$ to a decimal.

$$\begin{array}{l} 3)\underline{2.0} \\ \;\;\underline{.66666,}\ \&c. \end{array}$$

Ex. 5. Reduce $\frac{5}{6}$ to a decimal.

$$\begin{array}{r} 6\,)\,5.0 \\ \hline .8333, \text{ \&c.} \end{array}$$

Ex. 6. Reduce $\frac{7}{27}$ to a decimal.

$$\begin{array}{l} 27)\,7.0\,(\cdot259259259, \text{ \&c., \&c.} \\ \quad 160 \\ \quad\;\; 250 \\ \quad\;\;\; 70, \text{ \&c.} \end{array}$$

Ex. 7. Reduce $\frac{7}{22}$ to a decimal.

$$\begin{array}{l} 22)\,7.0\,(\cdot318181818, \text{ \&c., \&c.,} \\ \quad 40 \\ \quad\; 180 \\ \quad\;\; 40 \text{ \&c., \&c.} \end{array}$$

In these four examples the division does not terminate, but may be extended to any length whatever, and the quotient always has some continually recurring digit or digits, as 6 in Ex. 4, 259 in Ex. 6, and 18 in Ex. 7. Decimals of this kind are called *circulating*, *repeating* or *recurring* decimals, and the part repeated is called the *Period* or *Repetend.*

If the digits repeat immediately after the decimal point, as in examples 4 and 6, the decimal is called a *pure* circulator; if otherwise, as in examples 5 and 7, a *mixed* circulator.

When the repetend consists only of *one* digit, it is distinguished by a dot placed over it. Thus (Example 4.) $\frac{2}{3} = .\dot{6}$; (Example 5.) $\frac{5}{6} = \cdot8\dot{3}$. But when the repetend consists of more than 1 digit, it is distinguished by 2 dots over the first and last digits of the period. Thus, (Example 6.) $\frac{7}{27} = .\dot{2}5\dot{9}$; (Example 7.) $\frac{7}{22} = \cdot3\dot{1}\dot{8}$.

Ex. 8. Reduce $\frac{646}{25}$ to a decimal.

$$\frac{646}{25} = 25\tfrac{21}{25};$$

$$\begin{array}{l} 25)\,21.0\,(\cdot84 \\ \qquad 100 \end{array}$$

$$\therefore \frac{646}{25} = 25.84.$$

We first reduce the improper fraction to a mixed number, and then reduce the fractional part of it to a decimal. Also, when any fraction is to be reduced to a decimal, it should first be reduced to its lowest terms.

The reason of this operation will be evident, if we consider that the numerator of a vulgar fraction is understood to be divided by the denominator; and this division is actually performed when it is reduced to a decimal. And it is also evident that in every *proper* fraction, the quotient of the numerator by the denominator can contain *no* integer (otherwise it would be an *improper* fraction), and in some cases must have one or more cyphers prefixed to it. (Vid. Ex. 2.)

Ex. 9. Reduce $\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{8}$ to decimals. Ans. ·5, ·75, ·375.

Ex. 10. Reduce $\frac{1}{16}$, $\frac{13}{625}$, $\frac{106}{125}$, to decimals.
Ans. ·0625, ·0208, ·848.

Ex. 11. Reduce $\frac{9}{400}$, $\frac{17}{1250}$ to decimals. Ans. .0225, .0136.

Ex. 12. Reduce $\frac{11}{62\frac{1}{2}}$, and $6\frac{3}{2}$ of $\frac{1}{25} + \frac{17}{25}$ to decimals.
Ans. ·176 and ·98.

Ex. 13. Reduce $\frac{4}{9}$ and $\frac{1}{22}$ to decimals. Ans. $.\dot{4}$ and $\cdot 0\dot{4}\dot{5}$.

Ex. 14. Reduce $\frac{3}{7}$ and $\frac{1}{144}$ to decimals.
Ans. $\cdot\dot{4}2857\dot{1}$ and $\cdot 0069\dot{4}$.

Ex. 15. Reduce $\frac{11}{14}$ of $\frac{10}{13}$ to a decimal. Ans. $.\dot{6}0439\dot{5}$.

Ex. 16. Reduce $\frac{7}{60}$, $\frac{8}{27}$ and $\frac{7}{22}$ to decimals.
Ans. $.11\dot{6}$, $.\dot{2}9\dot{6}$, $.3\dot{1}\dot{8}$.

Ex. 17. Find the value of $\frac{21}{25} + \frac{170}{125} + \frac{15}{128} + \frac{13}{160}$.
Ans. 2.3984375.

Ex. 18. Reduce $\frac{13}{99}$, $\frac{17}{1375}$, $\frac{129}{55}$ to decimals.
Ans. $.\dot{1}\dot{3}$, $\cdot 012\dot{3}\dot{6}$, $2.3\dot{4}\dot{5}$.

Ex. 19. Find the value of $2 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots$ correct to 7 places of decimals. Ans. 2.7182818.

Ex. 20. Reduce $\frac{5}{6}+\frac{51}{55}+\frac{4}{21}+\frac{5}{11}+\frac{44}{45}$ to a decimal.
Ans. 3.3834, &c.

Ex. 21. Express $\frac{355}{113}$, $\frac{723}{668}$ in decimals to 5 places of figures. Ans. 3.14159 and 1.08232.

Ex. 22. Find the value of $2.5 - 6.002 + \frac{3}{\cdot 04}$. Ans. 71.498.

Ex. 23. Reduce $\frac{375}{999}$, $\frac{1}{28}$, $\frac{1}{81}$, $\frac{1}{216}$ to decimals.

Ans. $.\dot{3}7\dot{5}$, $\cdot 03\dot{5}7142\dot{8}$, $\cdot\dot{0}12345678\dot{9}$, $\cdot 004\dot{6}2\dot{9}$.

Ex. 24. Reduce $\frac{13}{60}$, $\frac{25}{99}$ and $\frac{1}{6}$ to decimals.

Ans. $\cdot 21\dot{6}$, $\cdot\dot{2}\dot{5}$ and $\cdot 1\dot{6}$.

Ex. 25. Reduce $\frac{1}{29}$ to a decimal.

Ans. $.\dot{0}34482758620689655172413793\dot{1}$.

ART. 53. Vulgar fractions whose denominators are 2 or 5, or any of their powers, may be expressed as *terminate* decimals. But if the denominators are not composed wholly of these, the division will continue, and consequently only an approximation to the value of the fraction can be obtained. But this approximation will differ very little from the true value, since if the division be carried on only to 4 places of decimals, the error cannot be greater than $\frac{1}{10000}$; and if the division be carried further, the error will become less.

The reason why the decimal will terminate if the denominator contain only powers of 2 and 5 as factors, is evident from the following consideration. Any power of 2 multiplied by the same power of 5 will produce an equal power of 10. Thus $2^3 \times 5^3 = 10^3$, for $2^3 \times 5^3 = 2 \times 5 \times 2 \times 5 \times 2 \times 5 = 10 \times 10 \times 10 = 10^3$. Hence if any power of 10 be divided by a like power of 2 or 5, the quotient will be an equal power of 5 or 2 and *exact*. Now in reducing vulgar fractions to decimals we annex cyphers

to the numerator; in other words, we make the numerator a multiple of some power of 10, and therefore whatever be the power of 2 or 5 in the denominator, the division will terminate, since we may always multiply the numerator by the *same* power of 10. (Vid. Algebra Art. 56.)

ART. 54. The reason why the decimal will not terminate if the denominator contains any other factor than 2 and 5 and their powers, is that the numerator is always made a multiple of 10 before division, and the only divisors of 10 are 2 and 5.

ART. 55. Also, whenever a decimal does not terminate it must *repeat*. For since we continually annex the *same* figure, viz. a cypher to the remainder, if any former remainder is repeated, the quotient will be repeated. Now some former remainder *must* be repeated; for all the remainders must be less than the denominator, and the number of *different* remainders must be less than the denominator. Since therefore the remainder must repeat, the quotient must also repeat. In Example 25, the denominator is 29, and 28 remainders (the greatest possible number) have occurred before a former remainder is repeated.

ART. 56. To reduce a decimal to a vulgar fraction.

I. If the decimal be terminate. (Vid. Art. 46.)

$$\text{Thus } \cdot 75 = \frac{75}{100} = \frac{15}{20} = \frac{3}{4},$$

$$\text{and } \cdot 312 = \frac{312}{1000} = \frac{78}{250} = \frac{39}{125}.$$

In all these cases we have as many cyphers in the denominator as there are decimal places.

II. If the decimal be a *pure* circulator, it may be reduced to a vulgar fraction by putting the *period* in the numerator, and as many nines as there are digits in the period, in the denominator.

Ex. 1. $.\dot{3} = \frac{3}{9} = \frac{1}{3}$.

Ex. 2. $\dot{0}4\dot{5} = \frac{45}{999} = \frac{5}{111}$.

Ex. 3. $.\dot{3}7\dot{8} = \frac{378}{999} = \frac{42}{111} = \frac{14}{37}$.

The reason of this is evident from the following considerations. In Example 1,

the required fraction = .333 &c.

$\therefore$ 10 multiplied by the required fraction = 3.33, &c. Art. 47.

$\therefore$ 9 multiplied by the required fraction = 3, by subtracting;

$\therefore$ the required fraction $= \frac{3}{9} = \frac{1}{3}$.

Again, in Ex. 2. the required fraction = .045045, &c.

$\therefore$ 1000 multiplied by required fraction = 45.045, &c.

999 multiplied by required fraction = 45;

$\therefore$ required fraction $= \frac{45}{999} = \frac{5}{111}$.

Hence the above Rule. (Vid. Algebra, Art. 55.)

III. If the decimal be a *mixed* circulator, it may be reduced to a vulgar fraction by putting the decimal to the end of the period diminished by the digits which do not circulate, for the numerator; and as many nines as there are circulating digits, followed by as many cyphers as there are noncirculating digits for the denominator.

Ex. 1. $.11\dot{6} = \frac{116 - 11}{900} = \frac{105}{900} = \frac{21}{180} = \frac{7}{60}$;

Ex. 2. $.01\dot{6} = \frac{16 - 1}{900} = \frac{15}{900} = \frac{3}{180} = \frac{1}{60}$.

Ex. 3. $.3\dot{1}\dot{8} = \frac{318 - 3}{990} = \frac{315}{990} = \frac{63}{198} = \frac{7}{22}$.

The reason of this is evident from the following considerations. In Example 1,

the required fraction = .11666, &c.

$\therefore$ 1000 multiplied by required fraction = 116.66, &c.;

and 100 multiplied by required fraction = 11.666

$\therefore$ by subtraction, 900 multiplied by required fraction = 105;

$\therefore$ required fraction = $\frac{105}{900}$.

In Ex. 3, required fraction = .3181818, &c.;

$\therefore$ 1000 multiplied by required fraction = 318.18, &c.;

and 10 multiplied by required fraction = 3.18, &c.;

$\therefore$ 990 multiplied by required fraction = 315;

or required fraction is $\frac{315}{990}$.

Hence the above rule. (Vid. Algebra, Art. 55.)

Obs. It is evident, that if an integer be prefixed to the ecimal in any of these three cases, it will produce no alteration a the rule.

$$\text{Thus, } 3.75 = 3\tfrac{75}{100} = 3\tfrac{3}{4}.$$

$$3.\dot{0}4\dot{5} = 3\tfrac{45}{999} = 3\tfrac{5}{111}.$$

$$3.3\dot{1}\dot{8} = 3\tfrac{315}{990} = 3\tfrac{7}{22}.$$

Ex. 1. Express as vulgar fractions .5, .25, .75, .04, .375.

Ans. $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{25}, \frac{3}{8}$.

Ex. 2. Reduce ·1015625, ·071575, ·0075264 to vulgar ractions.

Ans. $\dfrac{13}{128}, \dfrac{2863}{40000}, \dfrac{588}{78125}$.

Ex. 3. Reduce $.\dot{2}$, $.\dot{2}\dot{7}$, $.\dot{2}9\dot{6}$, and $.\dot{9}$ to vulgar fractions.

Ans. $\dfrac{2}{9}, \dfrac{3}{11}, \dfrac{8}{27}$, 1.

Ex. 4. Reduce $.\dot{9}\dot{0}$, $.41\dot{6}$, $.5\dot{3}$, to vulgar fractions.

Ans. $\dfrac{10}{11}, \dfrac{5}{12}, \dfrac{8}{15}$.

Ex. 5. Reduce $.\dot{0}\dot{1}$, $.232\dot{1}4285\dot{7}$, and $.\dot{1}\dot{8}$ to vulgar fractions.

Ans. $\dfrac{1}{99}, \dfrac{13}{56}, \dfrac{2}{11}$.

Ex. 6. Reduce $.\dot{1}6\dot{2}$, $.\dot{7}6\dot{5}$, $.91\dot{2}$, $.8\dot{1}\dot{2}$ to vulgar fractions.

Ans. $\dfrac{6}{37}, \dfrac{85}{111}, \dfrac{821}{900}, \dfrac{134}{165}$.

Ex. 7. Reduce $.56\dot{8}\dot{1}$, $2.4\dot{1}\dot{8}$, $.1\dot{1}\dot{3}\dot{6}$ to vulgar fractions.

Ans. $\frac{25}{44}$, $2\frac{23}{55}$, $\frac{5}{44}$.

Ex. 8. Reduce $.041\dot{6}$, $1.1\dot{4}\dot{5}$, $.093\dot{1}\dot{8}$ to vulgar fractions.

Ans. $\frac{1}{24}$, $1\frac{8}{55}$, $\frac{41}{440}$.

ART. 57. The fundamental operations of Arithmetic in which circulating decimals occur, may generally be performed with sufficient accuracy, by retaining only a few digits of the period, and rejecting the rest as inconsiderable.

Ex. 1. Add together $.3752\dot{4}$, $.\dot{8}$, $.643$, $.7\dot{3}$ correct to four places of decimals.

$$\begin{array}{r} .375244 \\ .888888 \\ .643 \\ .733333 \\ \hline 2.640465 \end{array}$$

If accuracy be required to more places of decimals, we must take more digits; and we must always take two or more places than the number required to be accurate.

Ex. 2. Subtract $.8473\dot{8}$ from $.935\dot{6}$ to four places of decimals.

$$\begin{array}{r} .935666 \\ .847388 \\ \hline .088278 \end{array}$$

We can also first reduce these circulating decimals to vulgar fractions, and then add or subtract; and this is far preferable in multiplication and division.

$$\text{Thus, } .235\dot{6}\dot{2} + .\dot{2}72\dot{7} = \frac{23562 - 235}{99000} + \frac{2727}{9999},$$

$$= \frac{23327}{99000} + \frac{2727}{9999},$$

which can be added, and then reduced to a decimal.

And, as in the Arithmetic of terminate decimals, we continually annex cyphers to the right hand of the decimal, so in interminate decimals, we continually annex the *period* to the right hand.

Ex. 1. Add, subtract, multiply, and divide $.\dot{4}$ and $.2\dot{1}\dot{7}$.

Ans. $.6\dot{6}\dot{1}$, $.2\dot{2}\dot{7}$, ·09652 &c., 2.04651 &c.

Ex. 2. Find the value of $.\dot{3} - .\dot{0}\dot{9}$ and $37.2\dot{3} \times .2\dot{6}$.

Ans. $.\dot{2}\dot{4}$ and $9.92\dot{8}$.

Ex. 3. Divide $.\dot{3}$ by $.\dot{0}\dot{9}$ and ·042 by $·03\dot{6}$.

Ans. $3.\dot{6}$ and $1.\dot{1}4\dot{5}$.

Ex. 4. Find the value of $.\dot{3}\dot{4} - .\dot{1}2\dot{8}$ and $.\dot{9}4\dot{3} - ·6\dot{8}\dot{2}$.

Ans. $.\dot{2}1530\dot{6}$ and $·2611156\dot{6}\dot{1}$.

Ex. 5. Multiply .145 by $.4\dot{8}$ and 6.9 by $2.8\dot{4}$.

Ans. $.070\dot{8}$ and $19.62\dot{6}$.

Ex. 6. Divide $.3784\dot{5}$ by $.\dot{6}$ and $.\dot{2}8\dot{4}$ by $.\dot{1}\dot{6}$.

Ans. $.56768\dot{3}$ and $1.75\dot{9}0\dot{0}$.

It may also be observed that when we approximate to the value of any interminate decimal by taking only a few digits, it will be more accurate, if the first rejected digit be 5 or greater than 5, to add 1 to the last retained digit. Thus .94643 is more nearly = .9464285 than .94642 is. So ·12 is more nearly = ·119 than ·11 is. For $·12 = \frac{12}{100}$, $·11 = \frac{11}{100}$, and $·119 = \frac{119}{1000}$; and it is readily seen that $\frac{12}{100}$ is more nearly $= \frac{119}{1000}$ than $\frac{11}{100}$ is.

Art. 58. To reduce lower denominations to decimals of higher.

Annex a cypher to the lower denomination, and divide it by the value of the higher. When there are several denominations, begin with the term of the lowest denomination, and reduce it to a decimal of the next higher, prefixing to this result the term of the higher denomination, and so on.

Ex. 1. Reduce 5*s*. to the decimal of a £1.

$$20\,\underline{)\,5.0}$$
$$·25$$

Ex. 2. Reduce 8*s*. 9*d*. to the decimal of a £1.

$$\begin{array}{r} 12\underline{)\,9.0} \\ 20\underline{)\,8.75} \\ \cdot 4375 \end{array}$$

Ex. 3. Reduce 19*s*. $10\frac{1}{2}d$. to the decimal of a £1.

$$\begin{array}{r} 4\underline{)\,2.0} \\ 12\underline{)\,10\cdot 5} \\ 20\underline{)\,19\cdot 875} \\ \cdot 99375 \end{array}$$

Ex. 4. Reduce 5 cwt. 2 qrs. 21 lb. to the decimal of a ton.

$$\begin{array}{r} 28\underline{)\,21.0} \\ 4\underline{)\,2\cdot 75} \\ 20\underline{)\,5\cdot 6875} \\ \cdot 284375 \end{array}$$

The reason for this operation is the same as that in common Reduction—all quantities of a lower denomination being reduced to higher by dividing by as many of the lower as make one of the higher.

Thus in Ex. 3, 2 farthings $= \frac{2}{4}d. = .5d.$;

$$\therefore\ 10\tfrac{1}{2}d. = 10.5d.;$$

$$\therefore\ 10\tfrac{1}{2}d. = \frac{10.5}{12}s. = .875s.;$$

$$\therefore\ 19s.\ 10\tfrac{1}{2}d. = 19.875s.;$$

$$\therefore\ 19s.\ 10\tfrac{1}{2}d. = \frac{19.875}{20}£. = \cdot 99375,$$

which operation is conveniently performed as above.

Ex. 5. Reduce 3 qrs. 22 lb. to the decimal of a cwt.

$$28 = \left\{\begin{array}{r} 4\underline{)\,22.0} \\ 7\underline{)\,5.5} \end{array}\right.$$

$$\begin{array}{r} 4\underline{)\,3.785714285\ \&c.} \\ \cdot 946428571\ \&c. = \cdot 94643 \text{ nearly.} \end{array}$$

It is sufficient to retain 3 or 4 places of an interminate decimal, adding 1 if necessary to the last retained digit. (Art. 57.)

Ex. 6. Reduce $33\frac{1}{4}$ of 3 qrs. to the decimal of $3\frac{3}{4}$ tons.

Art. 42. Ex. 19, $33\frac{1}{4}$ of 3 qrs. $= \frac{133}{400}$ of $3\frac{3}{4}$ tons;

and this fraction reduced to a decimal is $\frac{1.33}{4} = {\cdot}3325$.

In these cases we first reduce one quantity to the *fraction* of the other, and then reduce this fraction to a decimal.

Ex. 7. Reduce 7*s.*, 9*s.*, 16*s.* to the decimal of a £1.

Ans. ·35, ·45, ·8.

Ex. 8. Reduce 9*s.* 6*d.* to the decimal of a £1., and 2 qrs. 14 lb. to the decimal of a cwt. Ans. ·475, ·625.

Ex. 9. Reduce 15*s.* $6\frac{3}{4}$*d.* and 6*s.* $10\frac{3}{4}$*d.* to the decimal of a £1. Ans. ·778125, ·344.

Ex. 10. Reduce 2 furlongs to the decimal of a league; and 52 days to the decimal of a year. Ans. $\cdot 08\dot{3}$, ·14247.

Ex. 11. Reduce 7, 8, 9 lbs. av. to the decimal of a cwt.

Ans. ·0625, $\cdot 0\dot{7}1428$, ·080357.

Ex. 12. Reduce 2*s.* 2*d.* and 7*s.* $4\frac{1}{2}$*d.* to the decimal of a £1.

Ans. $\cdot 108\dot{3}$, and ·36875.

Ex. 13. Reduce 5*s.* to the decimal of 13*s.* 4*d.* Ans. ·375.

Ex. 14. Reduce 8*s.* $11\frac{1}{4}$*d.* to the decimal of a guinea, and $\frac{2}{5}$ of a guinea to the decimal of a £1. Ans. ·4256, and ·42.

Ex. 15. Reduce 1*s.* $6\frac{3}{4}$*d.* to the decimal of a £1., and to the decimal of £5. Ans. ·078125 and ·015625.

Ex. 16. Reduce 7 oz. 4 dwts. to the decimal of a pound troy, and 9 oz. $2\frac{2}{7}$ dr. to the decimal of a pound avoirdupois.

Ans. .6 and $\cdot\dot{5}7142\dot{8}$.

Ex. 17. Reduce 3 qrs. 3 lbs. 1 oz. $12\frac{4}{9}$ drs. to the decimal of a cwt. Ans. $.\dot{7}$.

Ex. 18. Reduce 3 wks. 4 days 5 hrs. 6′. 7″. to the decimal of a month. Ans. $.900\dot{4}$.

Ex. 19. Reduce 15 hrs. 14 mins. 6 secs. to the decimal of 2 days, and a week to the decimal of a year.

Ans. ·3173958 and ·0192307.

Ex. 20. From $\frac{2}{5}$ of a guinea take $\frac{3}{4}$ of 7*s*. 6*d*., and reduce the result to the decimal of a moidore. Ans. $\cdot 102\dot{7}$.

Art. 59. To find the value of any decimal of a given quantity.

Multiply the given decimal by the value of the next lower denomination, and mark off as many decimal places from the product as there are in the multiplicand. The rest are integers of the lower denomination.

Ex. 1. Find the value of ·425 of a £1.

$$\begin{array}{r} \cdot 425\ \pounds. \\ 20 \\ \hline 8.500\ s. \\ 12 \\ \hline 6.000\ d. \end{array} \qquad \therefore\ \cdot 425\pounds. = 8s.\ 6d.$$

Ex. 2. Find the value of ·046875 lb. av.

$$\begin{array}{r} \cdot 046875\ \text{lb.} \\ 16 \\ \hline 281250 \\ 46875 \\ \hline \cdot 750000\ \text{ozs.} \\ 16 \\ \hline 450 \\ 75 \\ \hline 12\cdot 00 \end{array} \quad \text{drams.} \qquad \therefore\ \cdot 046875\ \text{lb.} = 12\ \text{drams.}$$

Ex. 3. Find the value of ·07 of £2. 10*s*.

$$\pounds 2.\ 10s = 50s.;$$

$$\therefore\ \cdot 07 \text{ of } \pounds 2.\ 10s = \cdot 07 \times 50s = 3.50s.$$

$$\begin{array}{r} 12 \\ \hline 6.00d \end{array}$$

$\therefore$ the value is 3*s*. 6*d*.

Here we reduce the compound quantity to a simple one, and then proceed according to the Rule.

Ex. 4. Find the value of $\cdot 1\dot{6}$ of £1.

$$\begin{array}{r} \cdot 16666\ \text{£}. \\ 20 \\ \hline 3.33320\ s. \\ 12 \\ \hline 3.99840\ d. \end{array}$$

$\therefore$ $.1\dot{6}$ of £1 $=$ 3*s*. 3.9*d*. or 3*s*. 4*d*.

Or $.1\dot{6} = \frac{1}{6}$;

$\therefore$ $.1\dot{6}$ of £1. $= \frac{20}{6}s = 3s.\ 4d.$

Ex. 5. Find the value of £·8323 and ·002084 lb. Troy.

Ans. 16*s*. 7¾*d*. and 12 grains.

Ex. 6. Find the value of .625 of a gallon, and ·0625 of a barrel of beer. Ans. 2 qts. 1 pt. and 2 gals. 1 qt.

Ex. 7. Find the value of ·622, ·839, ·365 of £1.

Ans. 12*s*. 5¼*d*., 16*s*. 9½*d*., 7*s*. 3¾*d*.

Ex. 8. Find the value of ·7365 of 6*s*. 8*d*.

Ans. 4*s*. 10*d*. 3.68*f*.

Ex. 9. Find the value of $\cdot 97291\dot{6}$ of £1. and ·16875 of £1.

Ans. 19*s*. 5½*d*. and 3*s*. 4½*d*.

Ex. 10. Find the value of ·9375 cwt. and £$.54\dot{3}$.

Ans. 3 qrs. 21 lbs. and 10*s*. 10*d*. 1⅗*f*.

Ex. 11. Find the value of ·4786 days, and £5.734.

Ans. 11 hrs. 29′. 11.04.″ and £5. 14*s*. [illegible]16*d*.

Ex. 12. Find the value of ·0474609375 of £10. 13*s*. 4*d*. and $\cdot 13\dot{8}$ of 3.5 moidores. Ans. 10*s*. 1½*d*. and 1*s*. 4⅔*d*.

Ex. 13. Find the value of $\cdot 089\dot{2}85714\dot{}$ of 7*s*. and $.4897\dot{9}428571\dot{}$ of 3½ wks. Ans. 7½*d*. and 1 wk. 4 dys. 23 hrs. 59′. 56″.

Ex. 14. Find the exact values of ·123 and $\cdot 12\dot{3}$ of £1.

Ans. 2*s*. 5*d*. 2.08*f*. and 2*s*. 5½*d*. ⅘.

Ex. 15. Find the value of £·009765. and ·625*s*.

Ans. 2¼*d*. and 7½*d*.

Ex. 16. Find the value of .176 of 1 fur. 36 pls. 2 yds. 5 in. and ·22 of 3 qrs. 15 lbs.

Ans. 13 pls. 2 yds. 1 ft. 4 in. and 21 lb. 12 oz. 7.68 drs.

Ex. 17. Find the value of .5*s* + .7 crowns + £·125.

Ans. 6*s*. 6*d*.

Ex. 18. Find the value of £·634375 + ·025 of 25*s*. + $\cdot 31\dot{6}$ of 30*s*.

Ans. £1. 2*s*. $9\frac{3}{4}d$.

Ex. 19. Find the value of ·75 of 6*s*. 8*d*. − 1.84375 of 4*s* + $3.979\dot{6}$ of 2*s*.

Ans. 5*s*. 7*d*.

Ex. 20. Find the value of $2.8680\dot{5}$ of 3*s* + $.8\dot{3}$ of 4*s* − 1.8 of 5*s*.

Ans. 2*s*. $11\frac{1}{4}d$.

RULE OF THREE.

Art. 60. Examples in the Rule of Three in Decimals, are worked by precisely the same methods as those in the Rule of Three of Integers, or Fractions.

Ex. 1. If an ell of cloth be worth 3*s*. 10*d*., what is the value of $19\frac{1}{4}$ yards.

$$1 \text{ ell is worth } 3s.\ 10d.;$$
$$\therefore 5 \text{ qrs.} \ldots\ldots\ 3s.\ 10d.;$$
$$\therefore 1 \text{ qr.} \ldots\ldots\ \frac{3s.\ 10d.}{5};$$
$$\therefore 1 \text{ yd.} \ldots\ldots\ \frac{4}{5} \times 3s.\ 10d.;$$
$$\therefore 19\tfrac{1}{4} \text{ yds. or } 19.25 \text{ is worth } 19.25 \times \frac{4}{5} \times 3s.\ 10d.;$$
$$\ldots\ldots\ldots\ldots\ldots\ 3.85 \times 4 \times 3s.\ 10d. \text{ or } 15.4 \times 3s.\ 10d.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \pounds 2.\ 19s.\ 0.4d.$$

Ex. 2. If $26\frac{1}{2}$ yds. cost £3. 16*s*. 3*d*., what will $32\frac{1}{4}$ yds. cost.

Ans. £4. 12*s*. $9\frac{1}{2}d$.

Ex. 3. If 1 oz. of silver costs 5*s*. 6*d*., what is the price of a tankard that weighs 1 lb. 10 oz. 10 dwt.

Ans. £6. 3*s*. 9*d*.

Ex. 4. Find a number which shall have to 4 the same ratio that 3.75 has to 3. Ans. 5.

Ex. 5. If 1 lb. cost ·5625 of 2*s*., what is the value of ·75 of 1 cwt. Ans. £4. 14*s*. 6*d*.

Ex. 6. If ·0625 lbs. cost ·458$\dot{3}$*s*., what is the value of ·075 of a ton. Ans. £61. 12*s*.

Ex. 7. If 1 oz of gold be worth £4.18958$\dot{3}$ what is the value of ·3682291$\dot{6}$ lbs. Ans. £18. 10*s*. 3.0140625*d*.

Ex. 8. How many revolutions will a carriage wheel, the diameter of which is 3 ft., make in a mile, the diameter of a circle being $\frac{1}{3.1416}$ of the circumference. Ans. $1621\frac{1}{4}$ nearly.

CHAPTER IV.

PRACTICE.

ART. 61. THIS is the name given to a particular class of Examples in the Rule of Three, *viz.* when one of the terms is an *unit.* Thus 'What is the value of 25 cwt. 2 qrs. at £3. 6*s.* 6*d.* per *cwt.*?' or enunciated according to the Rule of Three, If *one* cwt. cost £3. 6*s.* 6*d.*, what will be the value of 25 cwt. 2 qrs.? Questions of this kind, instead of being solved by the Rule of Three, are worked by *Practice,* an operation in which the four elementary rules are sometimes jointly, sometimes separately employed.

Ex. 1. Find the value of 7643 yards at 4*s.* per yard.

The common way of doing this would be to multiply the 7643 by 4*s.*, and then reduce the shillings in the product to pounds. This reduction is avoided by taking aliquot parts of a pound, as follows:

	£.	7643
4*s.*	$\frac{1}{5}$	£1528. 12*s.*

For since 4*s.* is $\frac{1}{5}$ of £1., we may take 7643 times 4*s.*, by taking 7643 fifth-pounds; or by taking a fifth of £7643. Considering then the 7643 as pounds, the remainder 3 after division will be £3., which reduced to shillings and divided by 5 gives 12*s.*

Ex. 2. Find the value of 1773 yards at 3*d.* per yard.

	£.	1773.
3*d.*	$\frac{1}{80}$	£22. 3*s.* 3*d.*

3*d.* is $\frac{1}{80}$ of £1., hence dividing by 80 we get the result.

Obs. It will amount to the same thing if we consider the remainder in each case as yards, and then find the value. Thus in Ex. 1., the remainder is 3, and the value 4*s.* per yard, which amounts to 12*s.* In Ex. 2., the remainder is 13, and the value 3*d.* per yard, which amounts to 3*s.* 3*d.*

Ex. 3. Find the value of 4753 at 13*s.*, and of 7543 at 14*s.*

$$\begin{array}{rl} & 4753 \\ & \underline{\quad 13} \\ & 14259 \\ & \underline{4753} \\ 20) & \underline{61789} \\ & \underline{\underline{£3089.\ 9s.}} \end{array} \qquad\qquad \begin{array}{rl} & 7543 \\ & \underline{\qquad 7} \\ 10) & \underline{52801} \\ & \underline{\underline{£5280.\ 2s.}} \end{array}$$

These Examples are more simply done by multiplication and reduction, because the rate is not an aliquot part of a pound. In the latter, we multiply by 7 and divide by 10, which is evidently the same as multiplying by 14 and dividing by 20. If we had 7543 at 19*s.*, the simplest way would be of subtracting the value at 1*s.* from the value at £1.

Ex. 4. Find the value of 3287 at $5\frac{1}{4}d.$, and 4573 at $2\frac{3}{4}d.$

$$\begin{array}{c|c|l} & & 3287 \\ & & \underline{\qquad 5} \\ & d & 16435d \\ \frac{1}{4} & \frac{1}{4} & \underline{\quad 821\frac{3}{4}} \\ & 12) & \underline{17256\frac{3}{4}d} \\ & 20) & \underline{1438 \quad 0 \quad \frac{3}{4}} \\ & & \underline{\underline{£71.\ 18s.\ 0\frac{3}{4}d.}} \end{array} \qquad\qquad \begin{array}{c|c|l} d & & 4573 \\ 2 & \frac{1}{6} & 762 \quad 2 \\ \frac{1}{2} & \frac{1}{4} & 190 \quad 6\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \underline{\ \ 95 \quad 3\frac{1}{4}} \\ & 20) & \underline{1047 \quad 11\frac{3}{4}} \\ & & \underline{\underline{£52.\ 7s.\ 11\frac{3}{4}d.}} \end{array}$$

In the former Example we consider them as pence, and multiply by 5, and then take the value of $\frac{1}{4}d.$ In the latter, we consider them as shillings, and take $2d = \frac{1}{6}s.$, $\frac{1}{2}d = \frac{1}{4}$ of $2d.$, and $\frac{1}{4} = \frac{1}{2}$ of $\frac{1}{2}d.$

Ex. 5. Find the value of 4258 at 17*s.* 3*d.*, and 5842 at 12*s.* $4\frac{1}{2}d.$

$$\begin{array}{c|c|l} & & 4258 \\ & & \underline{\quad\ 17} \\ & & 29806 \\ & & \underline{4258\ \ } \\ & & 72386 \\ 3d & \frac{1}{4} & \underline{\ \ 1064 \quad 6} \\ & 20) & \underline{73450. \quad 6} \\ & & \underline{\underline{£3672.\ 10s.\ 6d.}} \end{array} \qquad\qquad \begin{array}{c|c|l} & & 5482 \\ 10s & \frac{1}{2} & 2741 \\ 2s & \frac{1}{5} & \ \ 548 \quad 4 \\ 3d & \frac{1}{8} & \ \ \ 68 \quad 10 \quad 6 \\ 1\frac{1}{2}d & \frac{1}{2} & \underline{\ \ \ 34 \quad\ 5 \quad 3} \\ & & \underline{\underline{£3391.\ 19s.\ 9d.}} \end{array}$$

In the former Example we consider them as shillings, and multiply by 17, and then take 3*d.* the $\frac{1}{4}$*s.* In the latter, we consider them as pounds, and take 10*s* = $\frac{1}{2}$£., 2*s* = $\frac{1}{5}$ of 10*s.*, 3*d* = $\frac{1}{8}$ of 2*s.*, and 1$\frac{1}{2}$*d* = $\frac{1}{2}$ of 3*d.*

Ex. 6. Find the value of 3592 at £3. 12*s.* 8*d.*, and 543 at £2. 5*s.* 10$\frac{1}{2}$*d.*

		3592		
		3		
		10776		
12*s*	$\frac{1}{5}$	2155	4	
8*d*	$\frac{1}{18}$	119	14	8
		£13050.	18*s.*	8*d.*

		543		
		2		
		1086		
5*s.*	$\frac{1}{4}$	135	15	
10*d*	$\frac{1}{6}$	22	12	6
$\frac{1}{2}$*d*	$\frac{1}{20}$	1	2	7$\frac{1}{2}$
		£1245.	10*s.*	1$\frac{1}{2}$*d.*

In the latter Example we multiply by 2, and find the value of 543 at £2.: and then since 5*s* = $\frac{1}{4}$£., we have, dividing 543 by 4, the value of 543 at 5*s.*, and similarly we find the value at 10*d.* and $\frac{1}{2}$*d.* In the former example we take 12*s* = $\frac{1}{5}$ of £3., and therefore divide 10776 by 5. We might also have taken 10*s* = $\frac{1}{2}$£., 2*s* = $\frac{1}{5}$ of 10*s.*, and 8*d* = $\frac{1}{3}$ of 2*s.*; and then added as before.

Ex. 7. Find the value of 5 cwt. 2 qrs. 14 lbs. at £2. 5*s.* 6*d.* per cwt.

		£.	*s.*	*d.*
		2	5	6
				5
		11	7	6
2 qrs.	$\frac{1}{2}$	1	2	9
14 lbs.	$\frac{1}{4}$		5	8$\frac{1}{4}$
		£12.	15*s.*	11$\frac{1}{4}$*d*

Here we first multiply by 5, and get the value of 5 cwt.; then we take 2 qrs = $\frac{1}{2}$ of a cwt., and dividing £2. 5*s.* 6*d.* by 2, we get £1. 2*s.* 9*d.*, the price of 2 qrs.; and then since 14 lbs. = $\frac{1}{4}$ of 2 qrs., by dividing £1. 2*s.* 9*d.* by 4, we get the value of 14 lbs.; and the sum of these is the value of 5 cwt. 2 qrs. 14 lbs.

Obs. If it had been at so much per *quarter*, we should reduce the 5 cwt. 2 qrs. to quarters, and then multiply by them, and take 14 lb. = $\frac{1}{2}$ qr.

The above Example may also be worked as follows:

		5 at £2. 5s. 6d.	
		2	
		10	
5s.	$\frac{1}{4}$	1	5
6d.	$\frac{1}{10}$		2 6
		£11. 7s. 6d.	

		£.	s.	d.
		2	5	6
2 qrs.	$\frac{1}{2}$	1	2	9
14 lbs.	$\frac{1}{4}$		5	$8\frac{1}{4}$
		1	8	$5\frac{1}{4}$
		11	7	6
		£12	15	$11\frac{1}{4}$

Various artifices will suggest themselves according to the nature of the question, but it will *generally* be found easier to work them as in the first case of the last Example, especially when the multiplier is so simple.

Great care must be taken in adding up the several values, *not* to add in the top line unless in particular cases, as in 1 cwt. 2 qrs. 14 lbs. at £5. 2*s*. 6*d*. per cwt. Here we should add in the top line, since it is the value of 1 cwt. Also, it is not necessary to take aliquot parts of the *former* line always. Thus, if Ex. 7. had been 5 cwt. 2 qrs. 11 lbs., we should have taken 7 lbs. = $\frac{1}{8}$ of 2 qrs., and 4 lbs. = $\frac{1}{14}$ of 2 qrs. also, and then divided the value of the 2 qrs. both by 8 and 14.

Ex. 8. Find the value of 5704 at $\frac{1}{4}$*d.*; 4573 at $\frac{3}{4}$*d.*; 7243 at 4*d.*; and 9786 at 9*d.* Ans. £5. 18*s*. 10*d.*; £14. 5*s*. $9\frac{3}{4}$*d.*; £120. 14*s*. 4*d.*; £366. 19*s*. 6*d.*

Ex. 9. Find the value of 4856 at 11*d.*; 23754 at $\frac{1}{2}$*d.*; 72564 at $\frac{3}{4}$*d.* Ans. £222. 11*s*. 4*d.*; £49. 9*s*. 9*d.*; £226. 15*s*. 3*d.*

Ex. 10. Find the value of 3752 at $4\frac{1}{2}$*d.*; 2842 at $3\frac{3}{4}$*d.*; 3572 at $7\frac{1}{2}$*d.* Ans. £70. 7*s.*; £44. 8*s*. $1\frac{1}{2}$*d.*; £111. 12*s*. 6*d.*

Ex. 11. Find the value of 7250 at 1*s*. $2\frac{3}{4}$*d.*; and 3894 at 17*s*. 6*d.* Ans. £445. 11*s*. $5\frac{1}{2}$*d.*; £3407. 5*s.*

Ex. 12. Find the value of 1071 at 1*s*. 10*d.*; 1765 at 9*s*. 2*d.*; and 637900 at 5*s*. 4*d.* Ans. £98. 3*s*. 6*d.*; £808. 19*s*. 2*d.*; £170106. 13*s*. 4*d.*

Ex. 13. Find the value of 5271 at 1*s*. 3½*d*.; 3664 at 14*s*. 9*d*.; and 5628 at 2*s*. 11*d*.

Ans. £340. 8*s*. 4½*d*.; £2702. 4*s*.; and £820. 15*s*.

Ex. 14. Find the value of 7514 at 4*s*. 7*d*.; 3592 at £3. 12*s*. 8*d*.; and 2710 at 19*s*. 2½*d*.

Ans. £1721. 19*s*. 2*d*.; £13050. 18*s*. 8*d*.; £2602. 14*s*. 7*d*.

Ex. 15. Find the value of 3215 at £1. 17*s*.; 3683 at £2. 4*s*. 11*d*.; and 95 at £15. 14*s*. 7¼*d*.

Ans. £5947. 15*s*.; £8271. 8*s*. 1*d*.; £1494. 7*s*. 4¾*d*.

Ex. 16. Find the value of 142 at £1. 15*s*. 2¾*d*.; and 999 at £9. 19*s*. 11¾*d*. Ans. £250. 2*s*. 6½*d*.; £9988. 19*s*. 2¼*d*.

Ex. 17. Find the value of 25 cwt. 2 qrs. 14 lbs. at £3. 17*s*. 6*d*. per cwt.; and 84 cwt. 3 qrs. 11 lbs. at £1. 1*s*. 10*d*. per cwt. Ans. £99. 15*s*. 7½*d*.; and £92. 12*s*. 6$\frac{13}{56}$*d*.

Ex. 18. Find the value of 6 ozs. 18 dwts. 20 grs. at 7*s*. 9*d*. per oz.; and 13 yds. 1 ft. 7 in. at 9*s*. 4*d*. per foot.

Ans. £2. 13*s*. 9$\frac{23}{40}$*d*.; and £18. 18*s*. 9⅓*d*.

Ex. 19. What is the import duty on 45 tons 17 cwt. 2 qrs. of iron at £7. 18*s*. 4*d*. per ton? Ans. £363. 3*s*. 6½*d*.

Ex. 20. What is the yearly rent of 1200 acres 3 roods at £1. 8*s*. 6*d*. per acre? Ans. £1711. 1*s*. 4½*d*.

Ex. 21. What is the price of 5 cwt. 3 qrs. 18 lbs. at 3*s*. 6*d*. per lb.? Ans. £115. 17*s*. 0*d*.

Ex. 22. Find the amount of the wages of 6 labourers for 28½ days at 2*s*. 3*d*. each per day. Ans. £19. 4*s*. 9*d*.

Ex. 23. Find the value of 173¼ yds. at 5*s*. 3*d*. per yard.

Ans. £45. 9*s*. 6¾*d*.

Ex. 24. Find the cost of 2627 sacks at 7*s*. 8½*d*. per sack; and the yearly amount of a servant's wages at 2*s*. 3½*d*. per day.

Ans. £1012. 9*s*. 9½*d*.; and £41. 16*s*. 5½*d*.

Ex. 25. Find the value of 153½ yds. at 7*s*. 3*d*. per yard; and the value of 37 cwt. 3 qrs. 2 lbs. at £3. 14*s*. 7½*d*. per cwt.

Ans. £55. 12*s*. 10½*d*; and £140. 18*s*. 5$\frac{13}{112}$*d*.

Ex. 26. What is the price of 247 lbs. 15 ozs. at 8*s*. 3*d*. per lb.; and 72 cwt. 3 qrs. 17 lbs. at £1. 4*s*. 6*d*. per cwt.?

Ans. £102. 5*s*. $5\frac{13}{16}$*d*.; and £89. 6*s*. $1\frac{1}{8}$*d*.

Ex. 27. From 17 cwt. 2 qrs. 14 lbs. subtract 18 lbs. in every cwt. Ans. 14 cwt. 3 qrs. $4\frac{3}{4}$ lbs.

Ex. 28. What is the tax upon £745. 14*s*. 8*d*. at 3*s*. 6*d*. in the pound? Ans. £130. 10*s*. $0\frac{4}{5}$*d*.

Ex. 29. What is the dividend on £1710. 14*s*. 6*d*., at 13*s*. $4\frac{1}{2}$*d*. in the pound? Ans. £1144. 0*s*. $11\frac{29}{80}$*d*.

Ex. 30. An officer's pay is 12*s*. 3*d*. per day; what is that a year? Ans. £223. 11*s*. 3*d*.

CHAPTER V.

INTEREST AND DISCOUNT.

INTEREST.

Art. 62. Interest is the *consideration* paid for the use of money; or the premium which the borrower of a sum of money pays to the lender for its use.

The *rate* of interest is the sum paid for the use of a certain sum for a certain time, as of £100 for one year, in which case it is termed the rate *per cent., per annum.* Thus, if £4 be paid for the use of £100 for one year, the interest is said to be at the rate of 4 per cent., per annum; or, more briefly, at 4 per cent.

The sum lent is called the *principal;* and the aggregate of the principal, together with its interest for any time, is called the *amount* for that time.

When the principal alone produces interest, it is called *Simple* Interest; but when the interest, as soon as it becomes due, is added to the principal, and the amount produces interest, it is called *Compound* Interest.

Art. 63. To find the simple interest of any given sum, for a given time, at a given rate per cent., per annum.

Multiply the given sum or principal by the number of years and by the rate, and divide the product by 100; the quotient will be the interest required.

Ex. 1. Find the simple interest and amount of £375. for 3 years at 5 per cent. per annum.

$$
\begin{array}{rl}
\pounds 375 & = \text{the principal.} \\
3 & = \text{the number of years.} \\
\hline
1125 & \\
5 & = \text{rate per cent.} \\
\hline
100)\,5625 & \\
\hline
56.25 & \\
20 & \\
\hline
5.00 &
\end{array}
$$

∴ the interest is £56. 5*s*.
and amount = £375. + £56. 5*s* = £431. 5*s*.

The interest of £375. for 3 years is the same as the interest of 3 × 375, or £1125. for 1 year. Now the interest of £100. for 1 year is £5.; therefore the interest of £1125. for 1 year $= \frac{1125}{100} \times 5$, or $\frac{5625}{100} = \pounds 56.25 = \pounds 56.\ 5s.$

Ex. 2. Find the simple interest and amount of £537. 10*s*. for 3 years and 9 months at 4 per cent., per annum.

$$
\begin{array}{r|rrrl}
 & \pounds 537. & 10s. & & \\
 & & 3\frac{3}{4} & & = \text{number of yrs.} \\
\hline
 & 1612 & 10 & & \\
\frac{1}{2}\text{ yr.} & 268 & 15 & & \\
\frac{1}{4}\text{ yr.} & 134 & 7 & 6 & \\
\hline
 & 2015 & 12 & 6 & \\
 & & & 4 & = \text{rate per cent.} \\
\hline
100) & 8062\pounds & 10 & & \\
\hline
 & 80.62\pounds & \cdot 1s. & & \\
 & 20 & & & \\
\hline
 & 12\cdot 50 & & & \\
 & 12 & & & \\
\hline
 & 6.0 & & &
\end{array}
\qquad \text{or} \qquad
\begin{array}{r}
\pounds 537.5 \\
3.75 \\
\hline
26875 \\
37625 \\
16125 \\
\hline
2015.625 \\
4 \\
\hline
100)\,8062\cdot 5 \\
\hline
80.625 \\
20 \\
\hline
12\cdot 5 \\
12 \\
\hline
6.0
\end{array}
$$

∴ interest is £80. 12*s*. 6*d*. and amount £618. 2*s*. 6*d*.

Obs. Sometimes it will be found more simple to work the Examples by decimals, when the numbers are capable of being expressed as *finite* decimals, as in the above Example.

Ex. 3. Find the interest of £279. 13*s*. 8*d*. for 3½ years, at 5¼ per cent.

$$
\begin{array}{r|rrr}
 & \pounds 279. & 13s. & 8d. \\
 & & & 3\frac{1}{2} \\
\hline
 & 839 & 1 & 0 \\
\frac{1}{2} \text{ yr.} & 139 & 16 & 10 \\
\hline
 & 978 & 17 & 10 \\
 & & & 5\frac{1}{4} \\
\hline
 & 4894 & 9 & 2 \\
\text{for } \frac{1}{4} & 244 & 14 & 5\frac{1}{2} \\
\hline
100) & 5139 & 3 & 7\frac{1}{2} \\
\hline
 & 51.39\pounds. & \cdot 03s. & \cdot 075d. \\
 & 20 & & \\
\hline
 & 7.83 & & \\
 & 12 & & \\
\hline
 & 10\cdot 035 & &
\end{array}
$$

$\therefore$ the interest is £51. 7*s*. $10\frac{7}{200}d$.

Ex. 4. Find the interest of £197. 10*s*. for 120 days at $3\frac{1}{2}$ per cent.

$$\text{Here, the time} = \frac{120}{365} = \frac{24}{73} \text{ of a year.}$$

$$\pounds 197.\ 10s \times \frac{24}{73} = 197.5\pounds \times \frac{24}{73} = \frac{4740}{73}\pounds.$$

Multiplying this by the rate, we have

$$\frac{4740}{73} \times \frac{7}{2} = \frac{2370}{73} \times 7 = \frac{16590}{73};$$

and dividing by 100, we have

$$\frac{1659}{730}\pounds = \pounds 2.\ 5s.\ 5\tfrac{31}{73}d.$$

The same rule applies in this Example as in the former, and it will generally be found simpler (as in this case) not to perform the division till last.

Obs. In the calculation of interest, 12 months, 52 weeks, or 365 days are taken for a year. Also, in finding the number of days from one period to another, the day computed *from* is not reckoned, but the day computed *to* is reckoned.

Ex. 5. Find the interest of £320. from the 20th of May till the 11th of July at 5 per cent.

$$\text{Number of days} = 52 = \frac{52}{365} \text{ of 1 year.}$$

$$320 \times \frac{52}{365} = \frac{64 \times 52}{73};$$

and multiplying by the rate, and dividing by 100, we have

$$\frac{64 \times 52}{73} \times \frac{5}{100} = \frac{64 \times 52}{73 \times 20} = \frac{16 \times 52}{73 \times 5}$$

$$= \frac{832}{365}£ = £2.\ 5s.\ 7\tfrac{5}{73}d.$$

If the time had been from the 20th of May to the 11th of July, both inclusive, then the number of days would be 53.

All Examples in which the time consists both of years and days, as 2 years 35 days, will be worked in the same way. Thus, 2 years 35 days = $2\frac{35}{365} = 2\frac{7}{73}$ years. Also, if the rate be in parts of a £1., as £2. 4*s*. 8*d*., it must be reduced to the fraction or decimal of a £1. Thus, £2. 4*s*. 8*d* = $2\frac{7}{30}$£.

Ex. 6. Find the simple interest of £3467. for 1 year, at 5 per cent., per annum. Ans. £173. 7*s*.

Ex. 7. Find the simple interest and amount of £125. 6*s*. 8*d*. for 4 years, at 5 per cent. Ans. £25. 1*s*. 4*d*. and £150. 8*s*.

Ex. 8. Find the simple interest of £456. 11*s*. 8*d*. for $2\frac{1}{2}$ years, at 5 per cent.; and of £576. 18*s*. 9*d*. for $8\frac{1}{3}$ years, at 4 per cent. Ans. £57. 1*s*. $5\frac{1}{2}$.; and £192. 6*s*. 3*d*.

Ex. 9. Find the simple interest of £547. 2*s*. 6*d*. for $5\frac{1}{2}$ years, at 4 per cent.; and of £576. 2*s*. 6*d*. for $7\frac{1}{4}$ years, at $4\frac{1}{2}$ per cent. Ans. £120. 7*s*. 4*d*.; and £187. 19*s*. $2\frac{1}{2}$*d*.

Ex. 10. Find the simple interest of £554. 10*s*. for 3 months, at 4 per cent.; and of £737. 10*s*. for 6 months, at 5 per cent. Ans. £5. 10*s*. $10\frac{3}{4}$*d*.; and £18. 8*s*. 9*d*.

Ex. 11. Find the simple interest of £25. for $3\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent.; and of £500. 13*s*. 4*d*. for $2\frac{3}{4}$ years, at $2\frac{3}{4}$ per cent. Ans. £3. 18*s*. 9*d*.; and £37. 17*s*. $3\frac{1}{10}$*d*.

Ex. 12. Find the simple interest of £34675. for 1 day, and for 17 days, at 5 per cent. Ans. £4. 15*s*.; and £80. 15*s*.

Ex. 13. Find the simple interest of £304. 3*s*. 4*d*. for 8 days, at 4 per cent.; and of £584. 10*s*. for 42 days, at 5 per cent.
Ans. 5*s*. 4*d*.; and £3. 7*s*. 3*d*.

Ex. 14. Find the simple interest of £219. 18*s*. for 84 days, at $4\frac{1}{2}$ per cent.; and of £197. 10*s*. for 120 days, at $3\frac{1}{2}$ per cent.
Ans. £2. 5*s*. $6\frac{1}{2}$*d*.; and £2. 5*s*. $5\frac{1}{4}$*d*.

Ex. 15. Find the simple interest of £3996. 15*s*. for 4 years 225 days, at £2. 13*s*. 4*d*. per cent. Ans. £492. 0*s*. $4\frac{4}{5}$*d*.

Ex. 16. Find the amount of 500 guineas from March 4th, 1844, to July 28th, 1845, at 5 per cent. Ans. £561. 15*s*.

Ex. 17. Find the simple interest of £250. 12*s*. 6*d*. from March 26th, 1840, to October 30, 1842, both inclusive, at 3 per cent. Ans. £19. 10*s*. $11\frac{7}{10}$*d*.

Ex. 18. £170. is due the 12th of August, of which £54. is paid on the 18th of September, £56. on the 17th of October, and the balance on the 14th of November; how much interest is due, at 5 per cent. Ans. £1. 11*s*. $0\frac{1}{2}$*d*.

Ex. 19. £540. is due August 18th, 1772; of which £50. is paid March 19th, 1773; £25. December 19th, 1773; £25. September 23rd, 1774; £110. August 18th, 1775. Required the interest and balance due on November 11th, 1775; interest being at 5 per cent. Ans. The number of days are 218, 272, 278, 329, and 85. The balance due is £411. 4*s*. 2*d*., and interest £81. 4*s*. 2*d*; the respective interests being £16. 2*s*. 6*d*.; £19. 1*s*. 2*d*.; £19. 0*s*. 9*d*.; £22. 5*s*. 3*d*.; £4. 14*s*. 6*d*.

Ex. 20. A person having an income arising from £360. 10*s*. at $3\frac{1}{2}$ per cent., exchanges it for an income arising from £315. at 4 per cent.; what is his annual loss or gain? Loss = $4\frac{1}{5}$*d*.

Ex. 21. Find the simple interest of £63. 15*s*. 6*d*. for $1\frac{1}{4}$ year, at $4\frac{1}{2}$ per cent.; and of £963. 10*s*. 6*d*. for $\frac{3}{4}$ year, at $3\frac{1}{2}$ per cent. Ans. £3. 11*s*. $8\frac{3}{4}$*d*.; and £25. 5*s*. 10*d*.

Ex. 22. Find the simple interest of £125. 8*s*. 6*d* for $\frac{1}{2}$ year at $3\frac{1}{4}$ per cent.; and of £135. 7*s*. 6*d*. for $\frac{3}{4}$ year at 4 per cent.

Ans. £2. 0*s*. 9*d*.; and £4. 1*s*. $2\frac{1}{2}$*d*.

Ex. 23. Find the simple interest of £380. 4*s*. 2*d*. from the first of January to the end of April 1844, at $4\frac{1}{2}$ per cent.

Ans. £5. 12*s*. 6*d*.

Ex. 24. Find the simple interest of £225. 6*s*. 8*d*. for $\frac{1}{2}$ year, at £3. 5*s*. per cent.; and of £150. for 55 days, at $4\frac{3}{4}$ per cent.

Ans. £3. 13*s*. $2\frac{3}{4}$*d*.; and £1. 1*s*. $5\frac{1}{2}$*d*. $\frac{50}{73}$.

Ex. 25. An annuity of £50. is put out to interest immediately after each payment, what will it amount to in 7 years, allowing 5 per cent. simple interest. Ans. £402. 10*s*.

Art. 64. All Examples in Commission, Brokerage, and Insurance, may be worked in exactly the same way as those in simple interest for one year at a given rate.

Ex. 1. What is the commission on £728. at $2\frac{3}{4}$ per cent.

£728
$2\frac{3}{4}$
1456
for $\frac{1}{2}$ 364
for $\frac{1}{4}$ 182
100) 2002
20.02
20
.40
12
4·80
4
3.20

Ans. £20. 0*s*. $4\frac{3}{4}$*d*.

Ex. 2. What is the brokerage on £9125. at $\frac{1}{8}$ per cent.

£9125 × $\frac{1}{8}$ = £1140. 12*s*. 6*d*.

and dividing this by 100 we have

11.40£. ·12*s*. ·06*d*.
20
8·12
12
1·50
4
2.00

Ans. £11. 8*s*. $1\frac{1}{2}$*d*.

Ex. 3. What is the insurance on goods whose value is £2516. 10*s.* at $3\frac{1}{8}$ per cent.

$$
\begin{array}{rrrrr}
 & £2516. & 10s. & & \\
 & & 3\frac{1}{8} & & \\
\hline
 & 7549 & 10 & & \\
\text{for } \frac{1}{8} & 314 & 11 & 3 & \\
\hline
100) & 7864 & 1 & 3 & \\
\hline
 & 78.64£ & \cdot 01s. & & \cdot 03d. \\
 & 20 & & & \\
\hline
 & 12.81 & & & \\
 & 12 & & & \\
\hline
 & 9\cdot 75 & & & \\
 & 4 & & & \\
\hline
 & 3.00 & & &
\end{array}
$$

Ans. £78. 12*s.* $9\frac{3}{4}$*d.*

Ex. 4. Find the commission on £903. 6*s.* 8*d.*, at $3\frac{1}{2}$ per cent. Ans. £31. 12*s.* 4*d.*

Ex. 5. Find the brokerage on £1987. 5*s.*, at $\frac{7}{8}$ per cent. Ans. £17. 7*s.* $9\frac{9}{40}$*d.*

Ex. 6. Find the insurance on £3208. 17*s.* 1*d.*, at £2. 12*s.* per cent. Ans. £83. 8*s.* $7\frac{1}{4}$*d.*

Art. 65. To find the Compound Interest of any given sum, for a given time, at a given rate per cent., per annum.

Find the first year's interest, and add it to the original principal; then considering this as a new principal, find its interest, and add as before to the new principal, and so on for any length of time, till we have found the final principal. From this final principal deduct the original principal, and the result will give the compound interest required.

Ex. 1. Find the compound interest of £500. for 3 years, at 5 per cent.

$$
\begin{array}{rrl}
 & £500 & = \text{original principal} \\
 & 5 & \\
\hline
100) & 2500 & \\
\hline
 & £25 & = \text{first year's interest.}
\end{array}
$$

```
      525 = new principal for second year = 500 + 25,
        5
100)2625
     26·25
        20
      5.00 or interest in second year is £26. 5s.
      551  5 = principal for third year = £525 + £26. 5s.
           5
100)2756  5
    £27·56.  ·05s.
        20
     11·25
        12
      3.00 or interest in third year is £27. 11s. 3d.;
```

∴ final principal or amount equals

£551. 5s + £27. 11s. 3d = £578. 16s. 3d.

original principal = £500. 0s. 0d.;

∴ compound interest required = £78. 16s. 3d.

The reason of this is evident from the definition of Compound Interest.

Ex. 2. Find the compound interest of £317. 10s. for two years, at 3 per cent.

```
     £317. 10s. original principal
                3
100)952    10
   £9.52.  ·1s.
      20
   10.50
      12
    6.00 or interest for first year is £9. 10s. 6d.
  £327.  0s.  6d. = principal for second year,
                3
100)981   1   6
  £9.81 ·01s. ·06d.
     20
  16·21
     12
   2·58
      4
   2·32 or interest for second year is £9. 16s. 2d. 2.32f.
```

$\therefore$ amount = £327. 0*s*. 6*d* + £9. 16*s*. 2*d*. 2.32*f*.;
= £336. 16*s*. 8*d*. 2.32*f*.;

original principal = £317. 10*s*.;

$\therefore$ required compound interest = £19. 6*s*. 8*d*. 2.32*f*.

Ex. 3. Find the compound interest of £400. for $2\frac{1}{2}$ years at 6 per cent.

$$\begin{array}{rl}
 & £400 \\
 & \underline{\quad 6} \\
100) & \overline{2400} \\
 & \underline{24} = \text{interest for first year.} \\
 & £424 = \text{principal for second year,} \\
 & \underline{\quad 6} \\
100) & \overline{2544} \\
 & \underline{25.44} = £25.\ 8s.\ 9\tfrac{3}{5}d = \text{interest for second year.} \\
 & £449.\ 8s.\ 9\tfrac{3}{5}d = \text{principal for third year,} \\
 & \underline{\qquad\qquad 6} \\
100) & \overline{2696.12\ \ 9\tfrac{1}{5}} \\
 & £26{\cdot}96.\ {\cdot}12s.\ {\cdot}096d. \\
 & \underline{\quad 20} \\
 & 19{\cdot}32 \\
 & \underline{\quad 12} \\
 & \underline{3.936}
\end{array}$$

or interest for third year is £26. 19*s*. 3.936*d*.;

and interest for half the third year is £13. 9*s*. 7.968*d*.;

$\therefore$ amount required = £449. 8*s*. 9.6*d* + £13. 9*s*. 7.968*d*
= £462. 18*s*. 5.568*d*.;

$\therefore$ required compound interest = £62. 18*s*. 5.568*d*.

Obs. In many cases these Examples will be facilitated by using decimals throughout. Thus in the above Example,

principal for third year = £449.44.

$$\begin{array}{rl}
 & \underline{\qquad 6} \\
100) & \overline{2696.64} \\
 & 26.9664 \\
 & \underline{\qquad 20} \\
 & 19{\cdot}3280 \\
 & \underline{\qquad 12} \\
 & \underline{3{\cdot}936}
\end{array}$$

or interest for third year = £26. 19*s*. 3.936*d*. as before.

Whenever, as in the above Example, compound interest is required for a fractional number of years, as $2\frac{1}{2}$, we must first calculate the amount for 2 years, and then the amount of this for $\frac{1}{2}$ of a year, subtracting from this final amount the original principal.

Ex. 4. Find the amount, at compound interest, of £650. for 5 years at 5 per cent. Ans. £829. 11*s.* $7\frac{1}{2}$*d.*

Ex. 5. Find the amount and compound interest of £160. for 4 years, at 6 per cent. Ans. £201. 19*s.* 11*d.*; and £41. 19*s.* 11*d.*

Ex. 6. Find the amount of £95. 16*s.* 8*d.* for 2 years, at $2\frac{1}{2}$ per cent., compound interest. Ans. £100. 13*s.* $8\frac{3}{5}$*d.*

Ex. 7. Find the amount of £540. in 3 years, at 4 per cent., compound interest. Ans. £607. 8*s.* $6\frac{1}{4}$*d.*

Ex. 8. Find the amount of £130. in 3 years, at 5 per cent., compound interest. Ans. £150. 9*s.* $9\frac{9}{10}$*d.*

Ex. 9. Find the compound interest of £160. in $2\frac{1}{2}$ years, at $3\frac{1}{2}$ per cent. Ans. £14. 7*s.* $10\frac{3}{4}$*d.*

Ex. 10. What is the amount of £270. 10*s.* in 2 years, at 3 per cent., per annum, compound interest? Ans. £286. 19*s.* $5\frac{1}{2}$*d.*

Ex. 11. What is the amount of £550. for 3 years and 6 months, at 4 per cent., compound interest? Ans. £631. 0*s.* $11\frac{1}{2}$*d.*

Ex. 12. What is the compound interest of £300. for 3 years, at $2\frac{2}{3}$ per cent.? Ans. £24. 12*s.* $10\frac{362}{375}$*d.*

Ex. 13. What is the difference between the amount of £250. accumulating during 3 years, at 3 per cent., compound interest, and the amount of the same sum for the same period at 4 per cent., simple interest? Ans. £6. 16*s.* $4\frac{19}{50}$*d.*

Ex. 14. Find the compound interest of £1563. 19*s.* 8*d.* for $2\frac{3}{4}$ years at $3\frac{1}{2}$ per cent., per annum. Ans. £165. 7*s.* 1*d.*

Ex. 15. Find the difference between the amounts, at simple and compound interest, of £895. 16*s.* for 2 years, at $3\frac{1}{2}$ per cent. Ans. £1. 1*s.* $11\frac{1}{4}$*d.*

Ex. 16. Find the amount of £270. in $2\frac{1}{3}$ years, at $3\frac{1}{2}$ per cent., compound interest. Ans. £292. 12*s.* 1*d.*

Obs. The operation of finding the compound interest of any sum for a large number of years, as 30, would be very tedious according to the above method; but may be facilitated by the introduction of logarithms.

Art. 66. In the above Examples both in Simple and Compound Interest, we have supposed the Principal, Rate, and Time given to find the Interest or Amount; similar considerations however will enable us to find the time, principal, or rate, when the other three are given. The case which commonly occurs is to find the interest or amount—the other three being unusual, and in Compound Interest more difficult without a knowledge of logarithms and Algebra.

DISCOUNT.

Art. 67. Discount is the allowance made for the payment of a sum of money before it becomes due, and is usually calculated at a given rate per cent.

If the discount be subtracted from the sum, the remainder is termed its *present worth*, being that sum which with its interest would amount to the given sum at the time it becomes due. Hence the discount may also be defined to be the difference between the whole sum and its present worth.

Thus, if interest be at the rate of 5 per cent. per annum, £200. in 1 year will amount to £210.; in other words, the *present worth* of £210. due 1 year hence will be £200., and its discount = 210 - 200 = £10.

Art. 68. Hence we see the difference between interest and discount. The *interest* of £200. for a year, at 5 per cent. will be £10.; but the *discount* of £200. for a year, at 5 per cent. will be the interest of the *present worth* of £200.; and this being the interest of a less sum will of course be less than £10. In all cases therefore, the interest of any sum, for any time, at any rate, will be greater than the *discount* of the same sum, for the same time, at the same rate, because this *discount* is the *interest of a* less sum.

Obs. In practice it is sometimes usual to consider the discount to be the interest on the future debt itself, and not on the present worth. According to this calculation the present worth will be less than it ought correctly to be; or the payer of the sum of money will be the gainer. The term *discount* is also frequently applied to the deduction for ready-money payments upon goods purchased, irrespective of the *time* at which the debt would otherwise be paid. Thus, if a tradesman throws off a discount of 5 per cent., or a shilling in the pound, for the payment of an account of £25., the actual payment would be £25 − 25*s* = £23. 15*s*.

ART. 69. To find the present worth and discount of any given sum, for a given time, at a given rate, per cent. per annum.

Find the amount of £100. at simple interest, for the same time and rate. Multiply the given sum by 100 and divide the product by this amount of 100; the result will be the present worth. From the given sum subtract this present worth for the discount.

Ex. 1. Find the present worth and discount of £487. 10*s*. 8*d*. due 6 months hence, at 6 per cent. (Vid. Art. 63.)

The amount of £100. in 6 months, at 6 per cent., is £103.

```
        £487. 10s. 8d.
                   10
        ---------------
        4875   6   8
                   10
      -----------------
   103)48753   6   8 (£473. 6s. 8d = present worth.
       -----
        755
        ---
        343
        ---
         34
         20
        ---
   103) 686 (6
        ---
         68
         12
        ---
   103) 824 (8
```

∴ the discount = £487. 10*s*. 8*d* − £473. 6*s*. 8*d* = £14. 4*s*.

In this Example, the present worth of £103. due 6 months hence, at 6 per cent. is £100., therefore the present worth of £487. 10*s*. 8*d*. under the same circumstances will be $\frac{100}{103} \times$ £487. 10*s*. 8*d*.; and hence the above Rule.

If we have to find the discount only, multiply the given sum by the interest of £100., and divide the product by the amount of £100. In the above example

$$\begin{array}{rrrrl} & £487. & 10s. & 8d. & \text{given sum.} \\ & & & 3 & \text{interest of £100.} \\ \hline 103) & 1462 & 12 & 0 & (£14.\ 4s = \text{required discount.} \\ & 432 & & & \\ \hline & 20 & & & \\ & 20 & & & \\ \hline 103) & 412 & (4 & & \end{array}$$

If this discount be subtracted from £487. 10*s*. 8*d*. we shall obtain the present worth £473. 6*s*. 8*d*. as before.

Ex. 2. Find the present worth and discount of £440. 10*s*. due $1\frac{1}{4}$ year hence, at $3\frac{1}{2}$ per cent.

The interest of £100. in the same time is £4. 7*s*. 6*d*.

The amount .... £104. 7*s*. 6*d* = $104\frac{3}{8}$.

$$£440.\ 10s \times 100 = 44050\,;$$

$$\text{And } 44050 \div 104\tfrac{3}{8} = \frac{44050 \times 8}{835} = \frac{8810 \times 8}{167} = \frac{70480}{167}.$$

$$\begin{array}{rl} 167)\ 70480 & (£422.\ 0s.\ 8\tfrac{104}{167}d = \text{present worth.} \\ 368 & \\ \hline 340 & \\ \hline 6 & \\ 20 & \\ \hline 120 & \\ 12 & \\ \hline 167)\ 1440 & (8\tfrac{104}{167} \\ 104 & \end{array}$$

$\therefore$ the discount = £18. 9*s*. $3\frac{63}{167}$*d*.

Obs. In this, as in many other Examples, the remainder is put down as a fraction of a *penny*, instead of reducing it to farthings, and making the remainder a fraction of a *farthing*.

Ex. 3. Find the present worth of £357. 10*s*. due 9 months hence, at 5 per cent. Ans. £344. 11*s*. $6\frac{3}{4}$*d*.

Ex. 4. Find the discount of £312. for 292 days, at 5 per cent.; and of £1050. for 1 year, at 5 per cent.
Ans. £12.; and £50.

Ex. 5. Find the present worth and discount of £651. 13*s*. 4*d*. due 5 months hence, at $4\frac{1}{2}$ per cent.
Ans. £639. 13*s*. $5\frac{1}{4}$*d*.; and £11. 19*s*. $10\frac{3}{4}$*d*.

Ex. 6. Find the discount of £1336. 11*s*. 3*d*. due $3\frac{1}{2}$ years hence, at 5 per cent. Ans. £199. 1*s*. 3*d*.

Ex. 7. Find the present worth of £1000. 10*s*. to be paid at the end of 5 years 4 months, at $4\frac{1}{2}$ per cent. Ans. £806. 17*s*. $1\frac{5}{31}$*d*.

Ex. 8. Find the interest and discount of £125. 8*s*. 6*d*. for $\frac{1}{2}$ year, at $3\frac{1}{4}$ per cent. Ans. £2. 0*s*. 9*d*.; and £2. 0*s*. $1\frac{1}{4}$*d*.

Ex. 9. What is the discount of £340. due 5 months hence, at 4 per cent., per annum? Ans. £5. 11*s*. $5\frac{1}{2}$*d*.

Ex. 10. Find the present value of £75. due 17 months hence, at 4 per cent. Ans. £70. 19*s*. $6\frac{1}{2}$*d*.

Ex. 11. What sum will discharge a bill of £18. 9*s*., abatement for present payment being made at 10 per cent?
Ans. £16. 12*s*. $1\frac{1}{5}$*d*.

Ex. 12. What would be the ready-money payment of an amount of £27. 13*s*. 6*d*., discount at 5 per cent? Ans. £26. 6*s*.

Ex. 13. What ready-money will discharge a debt of £170. due 5 months hence, allowing discount at 8 per cent?
Ans. £164. 10*s*. $3\frac{3}{4}$*d*. $\frac{15}{31}$.

Ex. 14. Find the interest and discount on £150. for 55 days, at $4\frac{3}{4}$ per cent. Ans. £1. 1*s*. $5\frac{1}{2}$*d*. $\frac{50}{73}$; and £1. 1*s*. $3\frac{3}{4}$*d*.

Ex. 15. Find the discount of £55. 6*s*. $9\frac{3}{4}$*d*. due in 2 years, at $3\frac{1}{2}$ per cent., per annum. Ans. £3. 12*s*. $4\frac{3}{4}$*d*.

Ex. 16. What is the present worth of £120. payable as follows; viz. £50. at 3 months, £50. at 5 months, and the remainder at 8 months, allowing discount at the rate of 5 per cent? Ans. £117. 14*s*. 4*d*.

Ex. 17. Find the discount on £273. 4*s*. 6*d*. due 3 months hence, at 4½ per cent. Ans. £3. 0*s*. 9½*d*.

Ex. 18. What sum will discharge a debt of £56. 5*s*., an abatement of 15 per cent. being made for present payment?

Ans. £47. 16*s*. 3*d*.

Ex. 19. Find the accurate present worth of £100. due 1 year hence, at 5 per cent; and the present worth that would be paid by a banker. (Vid. Obs. Art. 68.)

Ans. £95. 4*s*. 9$\frac{1}{7}$*d*.; and £95.

Ex. 20. By how much does the interest of £1073. exceed the discount for 12 months, at 5 per cent? Ans. £2. 11*s*. 1$\frac{1}{7}$*d*.

Art. 70. The calculation of the present worth of any sum of money at *compound* interest would be much more difficult. The principle would be the same in both cases; and in this we should have to find a sum which, being improved at compound interest, would amount to the given sum in the given time. This would be done by finding the amount of £100. for the same time at the same rate, at compound interest, and then proceeding as in last Art. And it is evident that when compound interest is allowed, the present worth will be less than when simple interest only is allowed.

STOCKS.

Art. 71. This is the name given to money which has been lent to Government, at a particular rate of interest; and the different national stocks, or public funds, are named according to this interest. Thus, the stock which pays 3½ for every £100. to the holders, is called the 3½ per cents.; and so that which pays 3 for every £100. is called the 3 per cents.

Stock is mere nominal property, and may be bought, sold, or transferred, like any other kind of property. Also, since the value of money is continually varying, the price of £100. stock will be continually varying. If a person buy £300. stock of the 3 per cents., he will receive £9. per annum. It is only in

the particular case when the stocks are *at par* that £100. stock is worth £100. money; in all other cases it will be worth more or less than £100. money.

If the 3 per cent. stocks are said to be at 98, it means nothing more than that £98. money will purchase £100. stock in the 3 per cents.; in other words, that £98. *money* will bring in £3. per annum. So again, if the $3\frac{1}{2}$ per cents., be at £102., then a person must give £102. money for £100. stock in the $3\frac{1}{2}$ per cents., or £102. *money* will bring in £$3\frac{1}{2}$. per annum.

ART. 72. Ex. 1. Find the sum that will purchase £750. stock in the 3 per cents. at £82.

£82. money will purchase £100. stock;
$\therefore$ £41. .... £50. ....
$\therefore$ 41×15 .... 50×15 ....
or, £615. money will purchase £750. stock.

Ex. 2. If I lay out £2000. in the 3 per cent. consols, when they are at $88\frac{1}{2}$, what income shall I derive?

£$88\frac{1}{2}$. money will bring in £3. per annum;
$\therefore$ $\frac{177}{2}$£. ....
$\therefore$ £177. .... £6. ....
$\therefore$ £2000. money will bring in $6 \times \frac{2000}{177}$ or $\frac{12000}{177}$ per annum;
or £67. 15*s*. $11\frac{11}{59}$*d*.

Ex. 3. What is the value of £1743, 3 per cent. stock, at $82\frac{7}{8}$ per cent.?

The value of £100. stock is £$82\frac{7}{8}$ money, or $\frac{663}{8}$;
$\therefore$ .... £1. .... $\frac{663}{800}$£. money;
$\therefore$ .... £1743. .... $1743 \times \frac{663}{800}$;
or the value of £1743, 3 per cent. stock is £1444. 10*s*. $2\frac{1}{2}$*d*. $\frac{4}{5}$.

Ex. 4. The 3 per cents. being at 84, how much money must be invested in them to produce an income of £150?

£84. money, *i. e.* £100. stock, will bring in £3. per annum.
∴ £840. . . . . £30. . . . .
∴ £4200. . . . . £150. . . . .

Ex. 5. How much 4 per cent. stock can be purchased by the transfer of £1000. stock from the 3 per cents. at 72, to the 4 per cents. at 90?

£100. stock in the 3 per cents. is worth £72. money, and
in the 4 per cents. £90. money;
∴ £100. stock in the 3 per cents. is worth
$\frac{72}{90} \times 100$£. stock in the 4 per cents,
or £80. stock in the 4 per cents;
∴ £1000. stock in the 3 per cents. will purchase £800. stock
in the 4 per cents.

Ex. 6. When the 3 per cent. stocks are at $90\frac{5}{8}$, and the $3\frac{1}{2}$ per cents. at $97\frac{7}{8}$; in which may capital be invested to the greatest advantage?

In the 3 per cents. $90\frac{5}{8}$ money will bring in £3.;
∴ 725 money £24.;
∴ £1. money . . . . $\frac{24}{725}$£.
In the $3\frac{1}{2}$ per cents. $97\frac{7}{8}$ money . . . . $\frac{7}{2}$£.;
∴ 783 money . . . . 28£.;
∴ £1. money . . . . $\frac{28}{783}$£.

And since (as will be found) $\frac{28}{783}$£ is greater than $\frac{24}{725}$£; therefore the $3\frac{1}{2}$ per cents. will bring in the greater income, and be more advantageous than the 3 per cents.

Ex. 7. Required the sum that will purchase £820. stock in the 5 per cents. at 108. Ans. £885. 12*s.*

Ex. 8. How much 3 per cent. stock when the funds are at $75\frac{1}{2}$., can be bought for £800.? Ans. £1059. 12*s.* $0\frac{1}{2}$*d.*

Ex. 9. The 4 per cents. being at $82\frac{1}{8}$, what must be given for £1000. stock? Ans. £821. 5*s.*

Ex. 10. What must the 3 per cents. be at, that a purchaser may get 4 per cent. for his money? Ans. 75.

Ex. 11. How much 3 per cent. stock at $83\frac{3}{4}$ can be purchased by £500.? Ans. £597. 0*s.* $3\frac{1}{2}d.$

Ex. 12. A person buys £500. stock at $98\frac{3}{4}$, and sells again at 103; what does he gain by the transaction? Ans. £21. 5*s.*

Ex. 13. If £1000. be laid out in the purchase of 3 per cent. consols at $81\frac{5}{8}$, at what price must the stock be sold to produce a gain of £100? Ans. $89\frac{63}{80}$.

Ex. 14. When £100. stock may be purchased in the 3 per cents. for $89\frac{1}{2}$, at what rate may the same quantity of stock be purchased in the $3\frac{1}{2}$ per cents. with equal advantage?
Ans. £104. 8*s.* 4*d.*

Ex. 15. A person invests £1000. in the 3 per cents. at $88\frac{3}{8}$; what will be the amount of his half-yearly dividends?
Ans. £16. 19*s.* $5\frac{1}{2}d.$

Ex. 16. A person transfers £11000. from the 4 per cents. at 92 to the 5 per cents. at 110; what is the difference in his income? Ans. £20.

Ex. 17. A person invests £3003. in the $2\frac{1}{2}$ per cents. at $49\frac{1}{2}$; what is his income? Ans. £151. 13*s.* 4*d.*

Ex. 18. At what rate per cent. will a person receive interest who invests his capital in the 3 per cents at 91?
Ans. £3. 5*s.* $11\frac{19}{91}d.$

Ex. 19. A person transfers £1000. stock from the 4 per cents. at 90 to the 3 per cents. at 72; how much of the latter stock will he hold, and what will be the alteration in his income? Ans. £1250.; and £2. 10*s.* less.

Ex. 20. *A* invests £1000. in the 3 per cents. at 84, and *B* the same sum in the 4 per cents. at 110; find their respective incomes, and the difference between them.
Ans. *A*'s £35. 14*s.* $3\frac{1}{4}d.$; *B*'s £36. 7*s.* $3\frac{1}{4}d.$; and difference, 13*s.*

Ex. 21. How much stock can be purchased for £2500. in the $3\frac{1}{2}$ per cents. at $100\frac{3}{4}$, and what income will be derived from it? Ans. £2481. 7*s*. $9\frac{1}{4}$*d*.; and £86. 16*s*. $11\frac{1}{2}$*d*.

Ex. 22. How much stock must be bought at $88\frac{1}{2}$ per cent., in order that by selling out at $90\frac{1}{4}$, 20 guineas may be gained? Ans. £1200.

Ex. 23. Find the value of a legacy of £2000. stock in the 3 per cents. at $92\frac{1}{8}$, the legacy being subject to a duty of 10 per cent. Ans. £1953. 17*s*. 4*d*.

Ex. 24. What amount of stock in the $3\frac{1}{2}$ per cents., will produce the same income as £3560. 18*s*. in the 3 per cent.? Ans. £3052. 4*s*.

Ex. 25. What would be the difference in annual income from investing £3450. in the 4 per cents. at 92, and the $3\frac{1}{3}$ per cents. at 69? Ans. £16. 13*s*. 4*d*.

Ex. 26. How much stock at $90\frac{3}{8}$ will £2353. purchase, brokerage being $\frac{1}{8}$ per cent.? Ans. £2600.

Ex. 27. Find the value of £1250. 4 per cents. at $79\frac{1}{4}$, brokerage being $\frac{1}{8}$ per cent. Ans. £992. 3*s*. 9*d*.

Obs. Stock is generally transferred by means of a broker, who charges a certain per centage on every £100. *stock*. The brokerage is added to the price of a purchase, but deducted from a sale of stock.

CHAPTER VI.

EXTRACTION OF SQUARE AND CUBE ROOTS.

SQUARE ROOT.

ART. 73. To extract the square root of any number, is to find that number, which when multiplied by itself will equal the given number. Thus the square root of 4 is 2, because $2 \times 2 = 4$, and the square root of 25 is 5. It is not always the case that the square root of a number can be found exactly. Thus, no number multiplied by itself will give 62, in other words, the square root of 62 cannot be found exactly.

The following signs are sometimes made use of in the extraction of roots. Thus, $\sqrt[2]{25}$, or $\sqrt{25}$ (without the exponent 2), signifies the square root of 25. Similarly, $\sqrt[3]{27}$ signifies the cube root of 27, and $\sqrt[4]{64}$, the fourth root of 64. These are also sometimes written $(25)^{\frac{1}{2}}$, $(27)^{\frac{1}{3}}$, $(64)^{\frac{1}{4}}$, with *fractional* exponents. Also $(25)^2$, $(27)^3$, $(64)^4$, with integral exponents, mean respectively the square of 25, or 25×25, the cube of 27, or $27 \times 27 \times 27$, and the fourth power of 64, or $64 \times 64 \times 64 \times 64$.

Thus, $5^2 = 25$, and $\sqrt{25}$ or $(25)^{\frac{1}{2}} = 5$.

ART. 74. Proof of the rule for pointing in square root.

Since the square root of 1 is 1,
.... 100 is 10,
.... 10000 is 100,
.... 1000000 is 1000, &c.

we see that the square root of a number of fewer than 3 figures consists only of 1 figure; that of a number of more than 2 and less than 5 figures, of 2 figures; that of a number of more than 4 and less than 7 figures, of 3 figures, and so on. Hence,

if a point be placed over every alternate figure, beginning at the unit's place, the number of such points will be the same as the number of figures in the square root.

So also in decimals,

since the square root of ·01 is ·1,
.... ·0001 is ·01,
.... ·000001 is ·001, &c.

we see that if the decimal be made to have an *even* number of places, and then the pointing made from the place of units towards the *right* hand over every alternate figure, the number of such points will be the number of decimal places in the root.

ART. 75. In the pointing of integers, we begin at the unit's place, and point to the *left;* in pointing decimals, we begin at the second decimal place, and point to the *right,* always taking care that the number of decimal places be even. Thus, if we have to point ·41679, it will be $\cdot 4\dot{1}6\dot{7}9\dot{0}$, not $\cdot\dot{4}1\dot{6}7\dot{9}$; as it would have been if they were integers. Similarly, if we have to point $\cdot 41\dot{6}$, it will be $\cdot 4\dot{1}6\dot{6}$, annexing 6 in this case instead of a cypher. The reason of this will be evident by taking a simple case, as in the extraction of the square root of ·4. If no cypher be annexed, the square root of ·4 will be ·2, whereas $\cdot 2 \times \cdot 2 = \cdot 04$; but if a cypher be annexed (which will not alter the value of the decimal,) the square root will be approximately ·6, and $\cdot 6 \times \cdot 6 = \cdot 36 = \cdot 4$ nearly. The same would appear if we expressed ·4 as a vulgar fraction $\frac{4}{10}$, and then extracted the root of the numerator and denominator.

ART. 76. To extract the square root of a number.

Point the given number according to the above directions, dividing it into periods of two figures. Every period will give one figure in the root.

Find the nearest lesser root of the left hand period. Place the figure so found in the quotient, and its square under the first period. Subtract its square from the first period, and to the remainder bring down the next period for a resolvend.

Double the quotient for the first part of the divisor; find how often this is contained in the whole resolvend, excluding the unit's place; and place the figure so found both in the quotient, and on the right hand of the divisor, which completes the divisor.

Multiply this divisor by the last figure in the quotient, subtract the product from the resolvend, and to the remainder bring down the next period for a new resolvend, and proceed as before.

Obs. The reason of this rule can hardly be explained without a knowledge of Algebra. (Vid. Algebra, Art. 57.)

Ex. 1. Find the square root of 133225, and prove the result.

$$\begin{array}{rll}
 & 1\dot{3}3\dot{2}2\dot{5}\,(365 & \text{root} \\
3^2 = & 9 & \\ \cline{2-2}
\text{first divisor } 66) & 432 & \text{resolvend} \\
 & 396 & \\ \cline{2-2}
\text{second divisor } 725) & 3625 & \\
 & 3625 & \\ \hline
\end{array}
\qquad
\begin{array}{r}
365 \\
365 \\ \hline
1825 \\
2190 \\
1095 \\ \hline
133225 \\ \hline
\end{array}$$

These Examples may always be proved by squaring the root, and adding in the final remainder, if any.

In the above, when we double the figure 6 of the quotient, it is 12, and we therefore add 1 to the other 6, making it 72.

Ex. 2. Find the square root of 36372961.

$$\begin{array}{rl}
 & 3\dot{6}3\dot{7}2\dot{9}6\dot{1}\,(6031 \\
 & 36 \\ \cline{2-2}
1203) & 3729 \\
 & 3609 \\ \cline{2-2}
12061) & 12061 \\
 & 12061 \\ \hline
\end{array}$$

In this Example, when we double 6 for the first part of the divisor, it is 12, and this is not contained in 3, (the resolvend excluding the unit's place), therefore we put 0 in the quotient and on the right of the divisor, and bring down the next period 29.

Ex. 3. Find the square root of 190968, and prove the result.

```
    190968(436                          436
    16                                  436
83) 309                                2616
    249                               1308
866) 6068                            1744
     5196                            190096
      872 remainder                     872
                                     190968
```

In this Example, the remainder 872 is greater than the divisor 866. If however we had taken 7 for the last figure in the quotient, then the product of the quotient and divisor would have been greater than the resolvend.

Ex. 4. Find the square root of 7596796.

```
          7596796.000000(2756.228
          4
47)       359
          329
545)      3067
          2725
5506)     34296
          33036
55122)    126000
          110244
551242)   1575600
          1102484
5512448)  47311600
          44099584
           3212016
```

In this Example, where there is a remainder after every period of the given number is brought down, we continue the division as far as we please by annexing periods of cyphers. The same might have been done in Ex. 3.

Ex. 5. Find the square root of ·2916.

```
     ·2916(·54
     25
104) 416
     416
```

Ex. 6. Find the square root of 3.026.

$$
\begin{array}{r@{\,}l}
 & \dot{3}\cdot0\dot{2}6\dot{0}0\dot{0}0\dot{0} \;(1\cdot7395 \\
 & \underline{1} \\
27) & 202 \\
 & \underline{189} \\
343) & 1360 \\
 & \underline{1029} \\
3469) & 33100 \\
 & \underline{31221} \\
34785) & 187900 \\
 & \underline{173925} \\
 & 13975
\end{array}
$$

The square root of a vulgar fraction is found by extracting the square root of the numerator and of the denominator; or by reducing the fraction to a decimal, and then extracting the square root of the decimal, according to circumstances.

Thus, $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $\sqrt{\frac{16}{25}} = \frac{4}{5}$, $\sqrt{\frac{25}{4}} = \frac{5}{2}$, $\sqrt{2\frac{1}{2}} = \sqrt{2.5}$.

Ex. 7. Extract the square roots of 119025, and 273529.

Ans. 345, and 523.

Ex. 8. Extract the square roots of 441, 5184, and 106929.

Ans. 21, 72, and 327.

Ex. 9. Extract the square roots of 956484, and 7513081.

Ans. 978, and 2741.

Ex. 10. Extract the square roots of 72, and 2268741.

Ans. 8·4852.., and 1506·23..

Ex. 11. Extract the square roots of 22071204, and 3271·4207.

Ans. 4698, and 57·19..

Ex. 12. Extract the square roots of ·582169, and 4·372594.

Ans. ·763, and 2·091..

Ex. 13. Extract the square roots of ·0081, and ·00032754.

Ans. ·09, and ·01809..

Ex. 14. Extract the square roots of 64·853, and 5.

Ans. 8·053.., and 2·236..

Ex. 15. Extract the square roots of ·000625, and $\cdot\dot{4}$.

Ans. ·025, and $.\dot{6}$.

Ex. 16. Extract the square root of 295·84, and 534·5344.
Ans. 17·2, and 23·12.

Ex. 17. Extract the square root of ·0000018225, and 41605·800625. Ans. ·00135, and 203·975.

Ex. 18. Extract the square root of $\frac{2304}{5184}$, $\frac{2704}{4225}$, and $\frac{9216}{12544}$. Ans. $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$.

Ex. 19. Extract the square root of $\frac{275}{341}$, and $\frac{357}{476}$.
Ans. ·89802.., and ·86602..

Ex. 20. Extract the square root of $51\frac{21}{25}$, and $9\frac{43}{49}$.
Ans. $7\frac{1}{5}$, $3\frac{1}{7}$.

Ex. 21. Extract the square root of $85\frac{14}{15}$, and $6\frac{2}{3}$.
Ans. 9·27.., and 2·5819..

Ex. 22. Extract the square root of $7\frac{2}{5}$, and $\frac{\cdot 00405224}{\cdot 064}$.
Ans. 2·7202, and ·25162...

Ex. 23. Find the value of $\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} + \frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$, to 6 places of decimals. Ans. $\sqrt{2}$, or 1·414213.

Ex. 24. Find the square root of 3, $\frac{3}{5}$, $\frac{5}{3}$, and 1043·29.
Ans. 1·732, ·77459, 1·29099, and 32·3.

Ex. 25. The hypothenuse of a right-angled triangle is 75, and one of the sides 45; what is the other side? Ans. 60.

Ex. 26. Find the square root of
16489·1281, and 120888·68379025.
Ans. 128·41, and 347·6905.

CUBE ROOT.

ART. 77. To extract the cube root of any number, is to find that number, which when multiplied by itself *twice*, will equal the given number. Thus the cube root of 8 is 2, because $2 \times 2 \times 2 = 8$, and the cube root of 27 is 3. It is not always the

case that the cube root of a number can be found exactly. Thus, there is no exact cube root of 65, the nearest root being 4. Using the abbreviations, we have

$$2^3 = 8,\ (8)^{\frac{1}{3}} \text{ or } \sqrt[3]{8} = 2,\ \sqrt[3]{64} = 4, \text{ \&c. \&c.}$$

ART. 78. Proof of the rule for pointing in cube root.

Since the cube root of 1 is 1,

.... 1000 is 10,

.... 1000000 is 100, &c.

we see that the cube root of a number of fewer than 4 figures consists only of one figure; that of a number of more than 3 and less than 7, of 2 figures; and so on. Hence, if a point be placed over every third figure, beginning at the unit's place, the number of such points will be the same as the number of figures in the cube root.

So also in decimals, since the cube root of ·001 is ·1.

.... ·000001 is ·01, &c.

we see that if the decimal be made to have 3, 6, 9, &c. figures, and then the pointing be made from the place of units towards the *right* hand, over every third figure, the number of such points will be the number of decimal places in the root.

ART. 79. In the pointing of integers, we begin at the unit's place, and point to the *left;* in pointing decimals, we begin at the third decimal place, and point to the *right,* always taking care that the number of decimal places be a multiple of 3. Thus, if we have to point ·41, it will be $\cdot 41\dot{0}$; or ·46, it will be $\cdot 46\dot{6}$, annexing 6 in this case instead of a cypher. The reason of this is similar to that given in Art. 75.

ART. 80. To extract the cube root of a number.

Point the given number according to the above directions, dividing it into periods of three figures. Each period will give one figure in the root.

Find the nearest lesser root of the left hand period. Place the figure so found in the quotient, and its cube under the first

period. Subtract its cube from the first period, and to the remainder bring down the next period for a resolvend.

The divisor consists of 3 parts, which may be formed thus. The first part is found by multiplying the square of the quotient by 3, and annexing 2 cyphers to the product. Find how often this first part of the divisor is contained in the resolvend, and place the figure so found in the quotient. The second part of the divisor is found by multiplying the former quotient by 3, and the product by the figure now put in the quotient, and annexing a cypher to this last product. The third part of the divisor is found by squaring the figure now put in the quotient.

Place these 3 parts under one another, as in addition; and their sum will be the divisor complete.

Multiply the divisor thus completed by the figure last put in the quotient; subtract the product from the resolvend, and to the remainder bring down the following period for a new resolvend. (Vid. Algebra, Art. 58.)

Ex. 1. Find the cube root of 12812904, and prove the result.

	12812904(234 root.	234
	8	234
Divisor 1389)4812 resolvend.		936
	4167 product.	702
1st part = 1200	645904 new resolvend.	468
2nd part = 180	645904	54756
3rd part = 9		234
1st divisor = 1389	1st part = 158700	219024
	2nd part = 2760	164268
	3rd part = 16	109512
	2nd divisor = 161476	12812904

Note. If the first part of the divisor happen to be equal to or greater than the resolvend, place 0 in the quotient, annex 2 cyphers to the first part of the divisor, and to the resolvend bring down another period, as in Ex. 2.

Also, if the product of the quotient figure into the divisor

happen to be greater than the resolvend, we must place a lesser figure in the quotient.

Ex. 2. Find the cube root of 28·75.

	$2\dot{8}\cdot75\dot{0}00\dot{0}$ (3·06 root.
1st part = 270000	27
2nd part = 5400	)1750000 resolvend.
3rd part = 36	1652616 product.
Complete divisor = 275436	97384 remainder.

In the proof of this Example, after cubing 3·06 we must add in the remainder 97384.

We might have continued the operation farther by annexing periods of cyphers continually to the remainder.

The cube root of a vulgar fraction is found by extracting the cube root of its numerator and denominator; or by reducing the fraction to a decimal, and then extracting the cube root of the decimal. Thus, the cube root of $\frac{3}{4}$ = cube root of ·75, and $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$.

Ex. 3. Find the cube roots of 389017 and 99252847.

Ans. 73 and 463.

Ex. 4. Find the cube roots of 48228544, and 40353607.

Ans. 364, and 343.

Ex. 5. Find the cube roots of 1860867, and 673373097125.

Ans. 123, and 8765.

Ex. 6. Find the cube roots of 12·977875, and 122615.327232.

Ans. 2·35, and 49·68.

Ex. 7. Find the cube roots of 13824, and .001906624.

Ans. 24, and ·124.

Ex. 8. Find the cube roots of 102·87, and 33·230979637.

Ans. 4.685. . , and 3·215. . .

Ex. 9. Find the cube roots of 316872·91637, and 8365·427.

Ans. 68·17. . , and 20·3.

Ex. 10. Find the cube roots of 100, and 4.

Ans. 4·641. . , and 1·5874. . .

Ex. 11. Find the cube roots of 884·736, and ·007077888.

Ans. 9·6, and ·192.

Ex. 12. Find the cube roots of $\frac{250}{686}$ and $\frac{343}{512}$.

Ans. $\frac{5}{7}$, and $\frac{7}{8}$.

Ex. 13. Find the cube roots of $12\frac{19}{27}$, and $405\frac{28}{125}$.

Ans. $2\frac{1}{3}$, and $7\frac{2}{5}$.

Ex. 14. Find the cube roots of $\frac{4}{7}$, $\frac{2}{3}$, $7\frac{1}{5}$, and $8\frac{5}{7}$.

Ans. $\cdot829\ldots$, $\cdot873\ldots$, $1\cdot93\ldots$, and $2\cdot057\ldots$

Ex. 15. Find the cube roots of $\frac{5}{6}$, $\frac{82}{216}$, and $2+\sqrt{3}$.

Ans. $\cdot94103\ldots$, $\cdot72408\ldots$, and $1\cdot55$.

Ex. 16. Find the value of $\sqrt[3]{52034\frac{10}{27}}$, and $(1\cdot331)^{\frac{2}{3}}$.

Ans. $37\frac{1}{3}$, and $1\cdot21$.

ART. 81. The operation of finding the cube root of a number may frequently be facilitated by logarithms. Thus, to find the cube root of 10000000, having given that logarithm $2.15435 = \cdot\dot{3}$.

$$\log.\ 2\cdot15435 = \cdot\dot{3}\,;$$

$\therefore$ by the principles of logarithms

$$\log.\ 215\cdot435 = 2\cdot\dot{3} = 2\tfrac{3}{9} = 2\tfrac{1}{3} = \tfrac{7}{3}\,;$$

$$\therefore\ 215\cdot435 = 10^{\frac{7}{3}} = (10000000)^{\frac{1}{3}},$$

or the cube root of $10000000 = 215\cdot435$.

CHAPTER VII.

DUODECIMALS.

ART. 82. It has been before observed (Art. 12.) that we cannot generally multiply together concrete quantities of any kind, as shillings by shillings, ounces by shillings, &c. There is an exception however to this rule in the case of Square and Cubic measure, since the number of *square* feet in any rectangular surface may be found by multiplying together the number of *linear* feet in its length and breadth, and consequently the product of linear feet and inches into linear feet and inches may always be considered to represent some superficies. Also because the number of *cubic* feet in any solid may be found by multiplying together the number of *linear* feet in its length, breadth, and height; therefore the product of any three linear dimensions may always be considered to represent some solid. Hence we can attach a meaning to this operation, which we cannot do in the multiplication of shillings by shillings, or ounces by shillings, &c., &c.

The method of multiplying feet and inches together will be to reduce them to the same denomination, and then multiply them; and the product will be *square* feet, *square* inches &c., or *cubic* feet, *cubic* inches &c.

Multiply 15 ft. 7 in. by 11 ft. 11 in.

```
        15 ft.  7 in. =  187   linear inches
        11 ft. 11 in. =  143   ..........
                        ----
                         561
                        748
                       187
                     -------
                144) 26741   square inches
                   ---------------------
                  9) 185 sq. ft.  101 sq. in.
                    ---------------------------
                     20 sq. yds.  5 sq. ft.  101 sq. in.
```

Find the value of 18 ft. 9 in. × 13 ft. 4 in. × 8 ft. 4 in.

```
18 ft. 9 in. =  225   linear inches
13 ft. 4 in. =  160   ..........
               13500
               225
               36000    square inches
  8 ft. 4 in =      100  linear inches
        1728) 3600000   cubic inches
           27) 2083 cub. ft.  576 cub. in.
                 77 cub. yds.  4 cub. ft.  576 cub. in.
```

ART. 83. Duodecimals, or Cross Multiplication, is a short method of performing the above multiplications without reducing them to the same denomination. In this, the dimensions are commonly taken in feet, inches, and parts, decreasing from left to right in a twelve-fold proportion; the inches and parts are termed primes, seconds, thirds, &c., and are distinguished by accents placed to the right, above the numbers to which they belong. Thus 9 feet, 4 inches, 2 seconds, 3 thirds, is written 9 ft. $4' \; 2'' \; 3'''$.

The following is the rule for cross multiplication.

Under the multiplicand write the corresponding denominanations of the multiplier.

Multiply every term in the multiplicand, beginning at the lowest, by each term in the multiplier successively, beginning with the highest; divide each product which is not of the denomination of feet by 12; add the quotient to the next product, and place the remainder under the multiplicand when the denomination of the multiplier is feet, one place removed to the right when it is primes, two places when it is seconds, three when it is thirds, &c.

Add together the products, and the result will be obtained.

Ex. 1. Multiply 15 ft. 7 in. by 11 ft. 11 in.

15 ft.	$7'$		
11 ft.	$11'$		
171 sq. ft.	5		multiplying by 11 ft.
14 sq. ft.	$3'$	$5''$	multiplying by 11 in.
185 sq. ft.	$8'$	5	= 185 sq. ft. 101 sq. in., as in Art. 82.

In this result, the 8 is eight-twelfths of a foot, and the 5 is five *square* inches, therefore the number of square inches is $8 \times 12 + 5 = 101$.

Ex. 2. Find the value of 18 ft. 9 in. × 13 ft. 4 in. × 8 ft. 4 in.

18 ft.	$9'$		
13 ft.	$4'$		
243 sq. ft.	$9'$		
6 sq. ft.	$3'$	0	
250	0	0	
8	4		
2000	$0'$		
83	$4'$		
2083 cub. ft.	$4'$		= 2083 cub. ft. 576 cub. in., as in Art. 82.

In this result, the 4 is $\frac{4}{12}$ or $\frac{1}{3}$ of a cubic foot $= \frac{1728}{3}$ or 576 cubic inches.

Ex. 3. Multiply 8 ft. $6'$ $9''$ by 7 ft. $3'$ $3''$.

8 ft.	$6'$	$9''$			
7 ft.	$3'$	$3''$			
59	$11'$	$3''$			multiplying by 7 ft.
2	$1'$	$8''$	$3'''$		multiplying by $3'$
	$2'$	$1''$	$8'''$	$3''''$	multiplying by $3''$
62 s.ft.	$3'$	$0''$	$11'''$	$3''''$	= 62 sq. ft. 36 sq. in. $11'''$ $3''''$.

There will always be the same number of *accents* in the product as in the two simple factors. Thus, $3'' \times 9'' = 27''''$, or $2'''.\ 3''''$.

Obs. Many Examples also may be worked by vulgar fractions or decimals, or practice. Thus if we have to multiply 7 ft. 9 in. by 3 ft. 6 in., we may do it by fractions thus,

$$7\text{ ft. }9\text{ in.} = 7\tfrac{3}{4}\text{ ft.},\quad 3\text{ ft. }6\text{ in.} = 3\tfrac{1}{2}\text{ ft.}$$

$\therefore$ the product is $7\frac{3}{4} \times 3\frac{1}{2} = 27\frac{1}{8}$ sq. ft. = 27 sq. ft. 18 sq. in.

or by decimals thus,

$$7 \text{ ft. } 9 \text{ in.} = 7.75, \qquad 3 \text{ ft. } 6 \text{ in.} = 3\cdot5;$$

$\therefore$ the product $= 7.75 \times 3.5 = 27\cdot125$ sq. ft. $= 27$ sq. ft. 18 sq. in.

or by practice thus,

$$\begin{array}{r|rrr} & 7 \text{ ft.} & 9 \text{ in.} & \\ & & 3 & \\ \hline & 23 & 3 & \\ 6 \text{ in} = \frac{1}{2} & 3 & 10 & 6 \\ \hline & 27 & 1 & 6 \end{array} = 27 \text{ sq. ft. } 18 \text{ sq. in.}$$

Any of these methods may be employed, but decimals should be used only when the inches can be expressed as a *terminating* decimal of a foot.

ART. 84. The reason of the above Rule for Cross Multiplication will appear from the following considerations.

Feet $\times$ feet give square feet, thus, 2 ft. $\times$ 5 ft. $=$ 10 square feet.
Feet $\times$ primes give primes, thus, 2 ft. $\times \frac{5}{12} = \frac{10}{12} = 10'$
Feet $\times$ seconds give seconds, thus, 2 ft. $\times \frac{5}{144} = \frac{10}{144} = 10''$.
Primes $\times$ primes give seconds, thus, $\frac{2}{12} \times \frac{5}{12} = \frac{10}{144} = 10''$.
Primes $\times$ seconds give thirds, thus, $\frac{2}{12} \times \frac{5}{144} = \frac{10}{1728} = 10'''$.

And it is evident that seconds, being $\frac{1}{144}$ of a foot, are square inches; and thirds, being $\frac{1}{1728}$ of a foot, are cubic inches. Any result therefore in which we have square feet, primes, and seconds, can be immediately reduced to square feet, and square inches, since a prime $= \frac{1}{12}$ square foot, and a second $= \frac{1}{144}$ square foot; and any result in which we have cubic feet, primes, seconds, or thirds, can be immediately reduced to cubic feet, and cubic inches, since in this case a prime $= \frac{1}{12}$ cubic foot, a second $= \frac{1}{144}$ cubic foot, and a third $= \frac{1}{1728}$ cubic foot, or 1 cubic inch.

ART. 85. Cross Multiplication is frequently called Duodecimals. But it differs from the Duodecimal notation in this; that in the former, the feet are expressed in the *denary* or common scale of notation, and the primes, seconds, &c. in the *duodenary* scale; in the latter, the feet, primes, seconds, &c. are all expressed in the *duodenary* scale. It will be readily seen

that if the feet are less than 10, the notation will be the same in each case, but if 10 or greater than 10, it will be different. Also in the Duodecimal notation, the multiplication is performed like that in decimals, 10 and 11 being supposed digits of this scale, and the *twelves* are carried instead of the *tens*. Thus let it be required to multiply 9 ft. 8′ 7″ by 3 ft. 10′, according to the duodecimal notation.

These quantities expressed in the duodenary scale are

$$\begin{array}{r l} 9.87 & \\ \underline{3{\cdot}t,} & t \text{ being put for } 10. \\ 8{\cdot}11t & \\ \underline{25{\cdot}19} & \\ 31{\cdot}2tt & = 37 \text{ feet } 2'\ 10''\ 10'''. \end{array}$$

In this Example we multiply as usual, and carry 1 for every 12 to the next product, and so on, and in the addition we also carry 1 for every 12. The 31 represent 31 feet on the duodenary scale, or $3 \times 12 + 1 = 37$ feet in the common scale, and the 2 are 2′, the first 10 are seconds, and the second 10, thirds.

The same done by Cross Multiplication would be as follows:

$$\begin{array}{r r r r} 9 & 8' & 7'' & \\ 3 & 10' & & \\ \hline 29 & 1' & 9'' & \\ 8 & 1' & 1'' & 10''' \\ \hline 37 & 2' & 10'' & 10''' \end{array}$$

By comparing these two operations, we see that the first line in the former corresponds to the 2nd line in the latter, and the second line in the former (expressed in the common scale) to the first line of the latter, since 25 in the duodenary scale = 29 in the common scale. If the number of feet had been greater than 9, we should have had first to express them in the duodenary scale before commencing the multiplication according to the duodecimal notation. Cross multiplication cannot therefore, with propriety, be called Duodecimals.

Art. 86. Ex. 1. Multiply 8 ft. 5 in. by 4 ft. 7 in.

Ans. 38 sq. ft. 6′. 11″.

Ex. 2. Multiply 8 ft. 1 in. by 3 ft. 5 in. Ans. 27. 7′. 5″.

Ex. 3. Multiply 17 ft. 8 in. by 3 yds. Ans. 17sq.yds. 6 sq.ft.

Ex. 4. Multiply 75 ft. 7 in. by 9 ft. 8 in. Ans. 730. 7'. 8''.

Ex. 5. Multiply 259 ft. 2 in. by 48 ft. 11 in.

Ans. 12677. 6'. 10''.

Ex. 6. Multiply 321. 7'. 3''. by 9. 3'. 6''.

Ans. 2988. 2'. 10''. 4'''. 6''''.

Ex. 7. Multiply 18 ft. 3 in. by 8 ft. 6 in.

Ans. 17 sq. yds. 2 sq. ft. 18 sq. inches.

Ex. 8. Find the content of a solid whose length is 2 ft. 3 in., width 1 ft. 7 in., and depth 9 in. 4''. Ans. 2 cub. ft. 9'. 3''.

Ex. 9. What is the length of a room whose breadth is 11 ft. 11 in., and which it takes 17 sq. yds. 2 ft. 131 in. of carpet to cover. Ans. 13 ft. 1 in.

Ex. 10. Find the solid content of a cube whose side is 7 ft. 5 in. Ans. 15 cub. yds. 2 ft. 1673 inches.

Ex. 11. The depth of a canal is 7 ft. 3 in., the width 20 ft. 4 in., and the length 10 miles; how many feet of water will it contain? Ans. 7783600 cub. feet.

Ex. 12. What is the area of a floor, its length being 15 ft. 9'. 8''., and width 12 ft. 5'. Ans. 196 sq. ft. 3'. 4'''.

Ex. 13. The edges of a cube being 2 ft. 8 in.; what is its solid content? Ans. 18 cub. ft. 11'. 6''. 8'''.

Ex. 14. How many square feet of glass are there in four windows, each measuring 5 ft. 7' in height, and 3 ft. 5' in width?

Ans. 76 sq. ft. 3'. 8''.

Ex. 15. A straight plank is $3\frac{1}{2}$ inches thick, and $6\frac{1}{4}$ broad; what length must be cut off of this uniform thickness and breadth, so as to contain exactly 20 cubic feet of timber?

Ans. $1579\frac{3}{3}\frac{1}{5}$ inches.

Ex. 16. A cistern 6 feet deep, contains 100 cubic inches, and its base is a square; find a side of the square. Ans. 1·18 in.

Ex. 17. The sides of a rectangular field are 253 yards, and $\frac{1}{4}$ mile, how many acres does it contain? Ans. 23.

Ex. 18. A room is $20\frac{1}{2}$ feet long, and $16\frac{1}{4}$ yards wide; what length of carpet will it take $\frac{3}{4}$ yard wide? Ans. $148\frac{1}{18}$ yds.

Ex. 19. A room is 13 feet 4 inches long, and 12 feet 6 inches broad; find the expence of carpeting it at 4*s*. 6*d*. per square yard. Ans. £4. 3*s*. 4*d*.

Ex. 20. The bottom of a cistern contains 16 square feet 18 inches; how deep must it be to contain 164 gallons, when 1 gallon = $277\frac{1}{4}$ cubic inches? Ans. $19\frac{1}{2}\frac{3}{3}\frac{5}{2}\frac{1}{2}$ inches.

Ex. 21. Find the area of a room 15 feet 9 inches long, and 14 feet 7 inches broad. Ans. 229 feet 8 inches 3 parts.

Ex. 22. A carpenter has a board 18 inches long and 8 broad; if he make it into a square, what will be its side? Ans. 12 inches.

Ex. 23. Find the length of a cube in inches, each of whose faces contains 4 square feet. Ans. 24 inches.

Ex. 24. A plank is 3 inches broad and 2 thick; what length must be cut off to contain 8 feet of timber? Ans. 2304 in.

Ex. 25. What is the worth of 8 squares of glass, each measuring 4 feet 10 inches long, and 2 feet 11 inches broad, at $4\frac{1}{2}$*d*. per foot? Ans. £2. 2*s*. $3\frac{1}{2}$*d*.

Ex. 26. What is the price of a slab, whose length is 5 feet 7 inches, and breadth 1 foot 10 inches, at 6*s*. per square foot? Ans. £3. 1*s*. 5*d*.

Ex. 27. The length and breadth of a room are known within $\frac{1}{10}$ of an inch; find the error committed in finding the area. Ans. $\frac{2}{100}$ inch.

Ex. 28. What is the cost of ceiling a room whose length is 74 feet 9 inches, and width 11 ft. 6 in., at 3*s*. $10\frac{1}{2}$*d*. per yard? Ans. £18. 10*s*. $1\frac{1}{4}$*d*.

Ex. 29. How many square feet are there in the floor of a room which measures 18 feet by $15\frac{1}{2}$ feet? Ans. 279.

Ex. 30. A room is 16 feet 8 inches long, and 15 feet 9 inches broad; find the expence of carpeting it at 6*s*. 6*d*. a square yard. Ans. £9. 9*s*. 7*d*.

Ex. 31. Find the side of a cube containing 4 solid feet 1088 inches. Ans. 20 inches.

Ex. 32. Find the breadth of a room, the length of which is $17\frac{1}{2}$ ft. and the area $250\frac{5}{6}$ ft. Ans. 14 ft. 4 in.

Ex. 33. Find the cost of papering a room 97 ft. 8 in. about, and 9 ft. 10 in. high, at 2*s*. $8\frac{3}{4}$*d*. per yard. Ans. £14. 11*s*. $2\frac{1}{2}$*d*.

Ex. 34. What length must be cut off a board, which is $6\frac{3}{4}$ inches broad, that the area may contain a square foot?
Ans. 1 foot $9\frac{1}{3}$ inches.

Ex. 35. What must be the length of a room whose breadth is $15\frac{3}{4}$ ft. that the area may contain 23 square yards?
Ans. $13\frac{1}{7}$ ft.

Ex. 36. What length of carpet, $\frac{3}{4}$ yard wide, will cover a room $6\frac{1}{2}$ yards long by $5\frac{1}{4}$ wide? Ans. $45\frac{1}{2}$ yds.

Ex. 37. Find the area of a room 23 ft. 8 in. long, and 16 ft. 7 in. broad, and explain the meaning of each term of the result.
Ans. 392 sq. ft. 5'. 8''.

Ex. 38. How much cloth will cover a room whose length is 12 ft. 6in., and breadth 2 ft. 9 in.? What will it cost at 5*s*. 6*d*. per square yard? Ans. 34 sq. ft. 4'. 6''; and £1. 1*s*.

Ex. 39. Find the number of square feet in a floor, length $10\frac{2}{3}$ yards and breadth $5\frac{1}{3}$ yards; and the price of paving it at 2*s*. per square foot. Ans. 512; and £51. 4*s*.

Ex. 40. How many square feet of paper will cover the walls of a room 20 ft. 10 ins. long, 16 ft. broad, and 10 ft. 8 ins. high? Ans. 785 sq. ft. 9'. 4''.

MISCELLANEOUS EXAMPLES.

Ex. 1. According to the common notation what is signified by 10100101?

Ex. 2. Subtract 3629 from 4184, and explain the operations.

Ex. 3. Multiply 1840 by 362, and explain the process.

Ex. 4. Divide 221 by 17, and shew that the common method of 'long division' gives a correct answer.

Ex. 5. Subtract £19. 13*s.* $10\frac{3}{4}$*d.* from £24. 5*s.* $7\frac{1}{2}$*d.*, and multiply £1. 13*s.* $4\frac{1}{2}$*d.* by 571, explaining the operations.

Ex. 6. If a man's salary be £100. a year, how much is that per day?

Ex. 7. Find how many inches there are in 5 miles 1 furlong 151 yards.

Ex. 8. If a bankrupt pay 6*s.* 8*d.* in the pound, how much will a creditor receive whose debt is £140. 10*s.* 6*d.*?

Ex. 9. Divide 4 cwt. 3 qrs. 17 lb. into 18 equal parts.

Ex. 10. What does $\frac{2}{3}$ multiplied by $\frac{3}{4}$ signify? Simplify the following expression:

$$\frac{5}{7} \times \left\{100 - \frac{2}{3} \text{ of } 100 + \frac{7\frac{1}{3}}{2\frac{1}{4}}\right\}.$$

Ex. 11. Subtract $\frac{3}{7}$ of an hour from $\frac{2}{5}$ of a day, and find what fraction of a guinea is $\frac{2}{3}$ of a penny.

Ex. 12. Add together $2\frac{1}{3}$ and $17\frac{7}{12}$, and subtract $\frac{2}{3}$ of $\frac{119}{18}$ from the sum.

Ex. 13. Find the value of $\sqrt{\frac{2}{3}}$ and $\sqrt[3]{\frac{2}{3}}$.

Ex. 14. Multiply and divide 3·1 by ·0025.

Ex. 15. Reduce 5 yds. 2 qrs. 7 ins. to the decimal of a yard; and find the value of that length of ribbon, when a piece of 100 yards costs 7*s.* 6*d.*

Ex. 16. Find the value of $\sqrt{\frac{5\cdot04}{\cdot012}}$, and $2 + \frac{4}{100} + \frac{6}{30000} + 3\cdot1416$.

Ex. 17. Find the value of $\frac{1}{2} + \frac{1}{3 \times 2^3} + \frac{1}{5 \times 2^5} + \frac{1}{7 \times 2^7}$, and shew that the decimal ·304565321 is more nearly represented by ·30457 than by ·30456.

Ex. 18. If the coach-fare from Cambridge to London is 21*s.* inside, and 14*s.* outside, find the rate per mile in each case, the distance being 51 miles.

Ex. 19. How much per day will give £126. 14*s.* $6\frac{1}{2}$*d.* per year?

Ex. 20. Required the side of a square whose area equals that of a rectangle, the sides of the rectangle being 597 yards $2\frac{1}{3}$ feet, and 37 yards 1 foot.

Ex. 21. There is a plantation in the form of a hollow square; the length of each side externally is 252 yards, and the breadth 16 yards; find the area of the plantation.

Ex. 22. Find the cost of 5736 lbs. at 3*s*. $8\frac{1}{2}$*d*. per lb.; and of 5 acres 3 roods 10 poles at £47. 13*s*. per acre.

Ex. 23. How much tea worth 3*s*. 10*d*. per lb. must be given in exchange for 28 lbs. of sugar worth $9\frac{1}{4}$*d*. per lb.?

Ex. 24. How many square yards of carpet will cover the floor of a room 16 ft. 8 in. long, and 13 ft. 5 in. broad, leaving the hearth-stone, which is 3 ft. 11 in. long, and 1 ft. 7 in. broad, uncovered?

Ex. 25. A certain sum is invested in the 3 per cents. at 90, and in the $3\frac{1}{2}$ at 102, which will give the larger interest?

Ex. 26. If 5 men can reap a field whose length is 800 ft. and breadth 700 ft. in $3\frac{1}{2}$ days of 14 hours each, in how many days of 12 hours each can 7 men reap a field whose length is 1800 ft., and breadth 960 ft.?

Ex. 27. Find the number of feet, inches, and parts in the side of a square whose area is 14 ft. 11 ins.

Ex. 28. A person holds as much stock of a certain kind as is worth £879, the price being £97. 10*s*. for £100 stock; there is also another stock which sells for £88. 5*s*. per £100; how much of the latter stock ought he to receive in exchange for his property in the former?

Ex. 29. State the difference between interest and discount, and find the discount on £397. 6*s*. 8*d*. due 9 months hence, at 4 per cent.

Ex. 30. Required the amount at compound interest of £257. in 4 years, at $3\frac{1}{2}$ per cent.

Ex. 31. If 4 men can do a piece of work in 5 hours, and 6 boys require 7 hours to do it in, find how long 5 men and 5 boys, working together, would be about it.

Ex. 32. *A* met 2 beggars, *B* and *C*; and having $\frac{3\frac{7}{11}}{4\frac{2}{7}}$ of $\frac{10\frac{5}{7}}{7\frac{1}{2}}$ of $\frac{77}{540}$ of a moidore in his pocket, gave *B* $\frac{1}{7}$ of $\frac{3}{4}$ of that sum, and *C* $\frac{3}{5}$ of the remainder; what did each receive?

Ex. 33. What sum must be invested in the 3 per cent. stock, at $94\frac{1}{4}$, to yield an annual income of £500?

Ex. 34. What is the interest of £365. 4*s*. 3*d*. at $3\frac{1}{2}$ per cent. for 2 years and 6 weeks?

Ex. 35. Required the price of $217\frac{1}{4}$ yards of lace, at £2. 17*s*. $7\frac{1}{2}$*d*. per yard.

Ex. 36. Compare the values of $\frac{5}{6}$ of a guinea, and $\frac{7}{8}$ of £1.; and reduce 19 ft. 7 in. to the fraction of a mile.

Ex. 37. What sum would be saved annually, if the interest on a public debt of £4,000,000. were reduced from $3\frac{1}{2}$ per cent. to 3 per cent.? If in consequence the price of this stock fell from £101. to £$95\frac{3}{8}$, how much would the whole property of the fundholders be diminished?

Ex. 38. *A* and *B* can do a piece of work in 7 days, *A* and *C* in 9 days, and *B* and *C* in 10 days, in what time will *A* perform it alone?

Ex. 39. Take $7\frac{1}{3}$ parts out of 53 from £1. 10*s*., and find the value of $\frac{\frac{1}{7} \text{ of } (\frac{2}{5} + 3\frac{1}{6}) + 4}{83 + \frac{1}{3} \text{ of } 5\frac{1}{11}}$.

Ex. 40. Find the value of $\frac{365}{\cdot 18349}$ and $\frac{1}{2.7182818}$ to 6 places of decimals.

Ex. 41. Find the value of $\cdot 00\dot{9}$ and $(673\cdot 492)^{\frac{1}{3}}$.

Ex. 42. If a person accepts £247. 1*s*. 8*d*. as present payment for £252. 0*s*. 6*d*. due 4 months hence, at what annual rate of interest does he allow discount?

Ex. 43. A cubic inch of water weighs 252·458 grains; required the weight (in lbs.) of water in a full cistern, $10\frac{1}{3}$ ft. long, $5\frac{1}{4}$ ft. wide, and 11 inches deep.

Ex. 44. Which is the greater, the simple interest of £273. 10*s*., or the discount of £283. 5*s*. for 7 months, at $3\frac{1}{2}$ per cent.?

Ex. 45. By selling tea at 5*s*. 4*d*. a lb. a grocer clears $\frac{1}{8}$ of the money; he afterwards raises the price to 6*s*. a lb. What does he clear per cent. at the latter price?

Ex. 46. What is the difference between $7\frac{4}{5}$ and $7 \times \frac{4}{5}$? Divide $15\frac{3}{5}$ by $7\frac{4}{5}$, and add together $\dfrac{15\frac{3}{5}}{7\frac{4}{5}}$£., $\frac{1}{3}$ of £140. 10*s*. 6*d*., and $\frac{4}{5}$ of a guinea.

Ex. 47. Required the values of the following quantities to 5 places of decimals:

$$\frac{1}{10+\dfrac{1}{2+\dfrac{1}{30}}} \quad \text{and} \quad \frac{1}{2\left\{\dfrac{9}{11}+\dfrac{1}{3}\left(\dfrac{9}{11}\right)^3+\dfrac{1}{5}\left(\dfrac{9}{11}\right)^5\right\}}.$$

Ex. 48. A sum of money at simple interest, has in $4\frac{1}{2}$ years amounted to £735., the rate of interest being 5 per cent. per annum, what was the sum at first?

Ex. 49. What is the cost of £850. bank annuities, at $90\frac{3}{8}$ per cent., $\frac{1}{8}$th per cent. being paid for brokerage?

Ex. 50. If the carriage of 60 cwt. 20 miles cost £$14\frac{1}{2}$, what weight can I have carried 30 miles for £$5\frac{7}{16}$?

ALGEBRA.

CHAPTER I.

DEFINITIONS AND EXPLANATION OF ALGEBRAICAL SIGNS AND TERMS.

ART. 1. ALGEBRA is a general method of reasoning concerning the relations which magnitudes of every kind bear to each other in respect of quantity. The symbols which it employs to denote magnitudes are more general and more extensive in their application than those employed in Arithmetic; and from the great facility with which the various relations of magnitudes to one another, may be expressed by means of a few signs or characters, the application of Algebra to the solution of problems is much more extensive than that of common Arithmetic.

In Arithmetic there are ten characters which serve to denote all magnitudes whatever; in Algebra recourse is had to a more general mode of notation, and quantities of every kind are denoted by any characters whatever; but those commonly used are the letters of the alphabet, the first letters $a, b, c, \ldots$ denoting generally *known* quantities, and the last $v, x, y, \ldots$ *unknown* quantities.

ART. 2. The *signs* made use of in Algebra are certain marks invented to imply the operations of Addition, Subtraction, Multiplication, Division, Involution and Evolution.

The sign of *addition* + (plus) signifies, when placed between two quantities, that the latter is to be combined with the former by the operation of addition. Thus $a + b$ indicates that the quantity represented by b is to be added to that represented by a; and if a and b represent 9 and 5 respectively, then $a + b$ is equivalent to $9 + 5$ or 14, where the sign does not appear. Similarly, $a + b + c + d$ indicates that these four quantities are to be combined by the operation of addition; and $b + b + b$ indicates that these three equal quantities are to be added together, and the result is $3b$.

ART. 3. The sign of *subtraction* − (minus) signifies, when placed between two quantities, that the latter is understood to be subtracted from the former. Thus $a - b$ indicates that the quantity represented by b is to be subtracted from that represented by a; and if a and b represent 9 and 5 respectively, then $a - b$ is equivalent to $9 - 5$ or 4. Similarly, $a - b + c - d$ indicates that the sum of b and d is supposed to be subtracted from the sum of a and c; and $-a-a-a$ is equivalent to $-3a$.

If the two quantities to be added or subtracted contain two or more terms, as $a + b$ and $c + d + e$, it is common to inclose them in a *parenthesis* or *bracket;* thus $(c + d + e) - (a + b)$ indicates that the sum of a and b is to be subtracted from the sum of c, d and e. It is also sometimes written $\overline{c + d + e} - \overline{a + b}$, the parts being connected by a line above them called a *vinculum*.

When the sign + is prefixed to a symbol or number it is called a *positive* quantity, and when − is prefixed, a *negative* quantity. When no sign is prefixed, + is understood, or it is considered a *positive* quantity.

Quantities which have the same sign are said to have *like* signs. Thus $+a$ and $+b$ have *like* signs; $+a$ and $-c$ have *unlike* signs.

All expressions formed by the operations of addition or

subtraction are called *compound* quantities. Thus, $(b+c)$ $(a+b-d)$ are *compound* quantities, but a is a *simple* quantity.

ART. 4. The sign of *multiplication* $\times$ (into) signifies when placed between two quantities, that they are to be multiplied together. Thus $a \times b$ indicates the product of the quantities represented by a and b; and if a and b represent 9 and 5 respectively, then $a \times b$ is equivalent to 9×5 or 45. Similarly $a \times b \times c$ indicates the product of a, b and c; and $(a+b-c) \times (d+e-f) \times (g+h-l)$ represents the product of the three compound factors, $a+b-c$, $d+e-f$, and $g+h-l$.

The sign $\times$ is frequently omitted in algebra, and sometimes a full point is used instead of it. Thus $a \times b \times c$ is written $a \, . \, b \, . \, c$ or abc; and $a \times (c+d) \times (e-f)$ is written $a\,(c+d)\,(e-f)$.

A number prefixed to a letter is called a *numeral coefficient.* Thus in $3a$, 3 is the numeral coefficient of a. When no number is prefixed, the coefficient is unity. Thus the coefficient of a is 1. In the expressions $8xy$, $6\,azy$, $8x$ is the coefficient of y, and $6a$ the coefficient of zy.

ART. 5. The sign of *division* $\div$ (by) signifies, when placed between two quantities, that the former is supposed to be divided by the latter. Thus $a \div b$ indicates that the quantity represented by a is to be divided by that represented by b; and if a and b represent 9 and 3 respectively, then $a \div b$ is equivalent to $9 \div 3$ or 3. Similarly $(a+b-c) \div (d+e+f)$ indicates the result arising from the division of the former of these compound quantities by the latter. Instead however of this sign, the operation is generally expressed by writing the dividend over the divisor as a fraction, thus $\frac{a}{b}$, $\frac{a+b-c}{d+e+f}$.

ART. 6. $=$ (equal) is the sign of equality; $>$ signifies *greater than;* $<$ signifies *less than;* $\therefore$ signifies *therefore,* and $\because$ signifies *because.* Thus $a=b$, $a>b$, and $a<b$, signify respectively that a is *equal* to, *greater* than, and *less* than b.

ART. 7. The sign of *involution* is a small numeral, called

an index or exponent, placed above the line, a little to the *right* of the quantity to which it belongs. Thus, a^2 is called the *square* of a, or *a squared*, and $= a \times a$. Similarly, a^3 is called the *cube* of a, or *a cubed*, and $= a \times a \times a$; and a^m is called the m^{th} power of a, and $= a \times a \times a \times \ldots$ to m factors.

The index of a is unity, or $a = a^1$. Similarly with compound quantities: thus $(a + bx)^2$, $(a + bx)^3$, $(a + bx)^n$ denote the square, cube, and n^{th} power, of $a + bx$.

Also, the square of $\frac{1}{a} = \frac{1}{a^2}$, and is sometimes written a^{-2}.

.... cube .. $= \frac{1}{a^3}$, ... a^{-3}.

.... m^{th} power .. $= \frac{1}{a^m}$, ... a^{-m}.

and the square, cube, &c. of $\frac{a}{b}$ are written a^2b^{-2}, a^3b^{-3} &c.

Art. 8. The sign of *evolution* is $\sqrt{}$, and denotes that the root expressed by the numeral with it, is understood to be extracted. Thus $\sqrt[2]{a}$ or $\sqrt{a}$ expresses the *square* root of a, or that quantity which when multiplied by itself produces a. So $\sqrt[3]{a}$, $\sqrt[m]{a - b}$ express the cube root of a, and the m^{th} root of $a - b$ respectively, and $\sqrt{a^3}$ expresses the square root of *a cubed*. Instead however of this sign, the operations are generally expressed by means of *fractional indices*.

Thus, $\sqrt[2]{a}$ or $\sqrt{a}$ is frequently written $a^{\frac{1}{2}}$;

$\sqrt[3]{a}$ and $\sqrt[m]{a - b}$ .... $a^{\frac{1}{3}}$ and $(a - b)^{\frac{1}{m}}$,

So also $\sqrt{a^5}$, $\sqrt[m]{(a - b)^2}$, $\sqrt{a^{m-n}}$ are frequently written $a^{\frac{5}{2}}$, $(a - b)^{\frac{2}{m}}$, $a^{\frac{m-n}{2}}$.

Quantities like $\sqrt{2}$, $\sqrt{4ax^2}$, where the roots cannot be *accurately* extracted are called *surds*.

Quantities involving $\sqrt{-1}$, as $\sqrt{-a^2}$, $\sqrt{-xy}$, &c., are called *impossible* or *imaginary* quantities.

Art. 9. $\propto$ is the sign of Variation. Thus, if a varies as b, it is written $a \propto b$. ∞ signifies *infinity*.

Art. 10. A *monomial* is an expression consisting of one term only, as a, ab, $-abx$, &c.

A *binomial* is an expression consisting of two terms, as $a+b$, $x-y$, $3a-4y$, &c.

A *trinomial* is an expression consisting of three terms, as $a+b+c$, $x+y-z$, x^2-px+9, &c.

A *multinomial* is an expression consisting of any number of terms connected by the signs + and −.

Like quantities are those which consist of the same letters, as a, $2a$; ab, and $-5ab$; x^n and $-2x^n$, &c. But a, b; ab, and ac, &c., are *unlike quantities.*

Art. 11. Quantities are said to have the same dimensions, or degrees, when the sum of the exponents in each of the terms is the same, unity being considered the exponent of a simple symbol, as a, b, c. Such quantities are also said to be *homogeneous*. Thus, $a^2x - ax^2$, and $a^3 - 3a^2x + 3ax^2 - x^3$ are *homogeneous* expressions. Also, a, ax^2, $(a-x)^m$ are called quantities of 1, 3, and m dimensions; since the sum of the indices is 1, 3, and m respectively. A numerical coefficient does not affect the *dimensions* of a quantity, since it alters its magnitude only, and not its nature.

Art. 12. Quantities are said to be arranged according to the dimensions of any letter, when the indices of that letter occur in the order of their magnitudes, either increasing or decreasing. Thus, $a^2 - ax + x^2$ are arranged according to the *descending* powers of a, and *ascending* powers of x.

Art. 13. If four quantities, as a, b, c, d, are proportionals, they are expressed thus; $a : b :: c : d$, or $a : b = c : d$, which denotes that a is the same multiple, part or parts of b, that c is of d, and has the same signification as $\frac{a}{b} = \frac{c}{d}$.

ART. 14. The following Examples will illustrate the above definitions.

Ex. 1. Find the value of $a+b+c-d+\sqrt{abcd+a^4}$, when a, b, c, d, equal 1, 2, 3, 4.

$$a+b+c-d+\sqrt{abcd+a^4} = 1+2+3-4+\sqrt{1 \cdot 2 \cdot 3 \cdot 4+1}$$
$$= 2+\sqrt{24+1} = 2+\sqrt{25} = 2+5 = 7.$$

Ex. 2. On the same supposition, prove that

$$(12a-b)(25a-3c)(30a-5d) = 1600;$$

also, if $e=5$, $f=6$, shew that

$$a^2+b^2-(c+d)^2+3e^2-ef+\sqrt{ad} = 3;$$

and $\sqrt{2(a^2+ab+b^2)+3(2c-d)^2-be}+ac+f = 13.$

Ex. 3. Shew that $\sqrt{x+13}+\sqrt{x} = 13$ when $x = 36$;

and $\sqrt{4-\sqrt{x^4-x^2}}-x = -2$ when $x = 2\frac{1}{8}$.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF SIMPLE ALGEBRAICAL QUANTITIES.

ADDITION.

ART. 15. THE fundamental operations in Algebra are the same as in Arithmetic, viz. Addition, Subtraction, Multiplication, and Division, and from the combination of these four, all the others are derived.

The sum of two quantities in Algebra is expressed by simply connecting them with their proper signs. Thus the sum of a and b is $a+b$, the sum of c and $-d$ is $c-d$, of $-e$ and $-f$ is $-e-f$.

In the addition of quantities there are 3 cases.

(1) where the quantities are *like* with the same sign.

(2) .... different sign.

(3) .... *unlike* with any signs.

CASE I. Add together the coefficients of the quantities, prefix the common sign to the sum, and annex the letter or letters common to each term.

$$
\begin{array}{lr}
(1) & 7a \\
 & 3a \\
 & a \\
 & 2a \\
\hline
 & 13a
\end{array}
\qquad
\begin{array}{lr}
(2) & -2ax \\
 & -ax \\
 & -5ax \\
 & -12ax \\
\hline
 & -20ax
\end{array}
\qquad
\begin{array}{lrr}
(3) & 5a & -6b \\
 & 8a & -4b \\
 & 11a & -23b \\
\hline
 & 24a & -33b
\end{array}
$$

CASE II. Add the positive coefficients into one sum, and the negative ones into another; subtract the least of these sums from the greatest, prefix the sign of the greatest to the remainder, and annex the letters as in Case I.

$$
\begin{array}{lr}
(1) & 2ax \\
 & -ax \\
 & -3ax \\
 & 9ax \\
\hline
 & 7ax
\end{array}
\qquad
\begin{array}{lrr}
(2) & 6ab & +7 \\
 & -4ab & +9 \\
 & ab & -5 \\
 & 7ab & -13 \\
\hline
 & 10ab & -2
\end{array}
\qquad
\begin{array}{lrrrr}
(3) & a^2 & +2ax & -x^2 & -4a^2b \\
 & -2a^2 & +3ax & -4x^2 & +a^2b \\
 & 6a^2 & -5ax & +11x^2 & +4a^2b \\
\hline
 & 5a^2 & & +6x^2 & +a^2b
\end{array}
$$

In (1) the sum of the positive quantities is $11ax$, and of the negative $-4ax$, therefore the result is $11ax - 4ax = 7ax$, by the principles of Arithmetic; and similarly in the other two Examples.

Obs. Previous to commencing the addition, the like quantities should be arranged under one another.

CASE III. Put down the quantities in any order, and connect them with each other by their own proper signs.

$$(1)\quad \begin{array}{l} 2a \\ 3b \\ -4c \\ \hline 2a+3b-4c \end{array} \qquad\qquad (2)\quad \begin{array}{l} ax+2ay \\ b^2-3bz \\ \hline ax+2ay+b^2-3bz. \end{array}$$

Obs. If in any result two or more terms have a common coefficient, it is customary to enclose these terms in a vinculum, and place the common coefficient before it.

Thus, $2a^2 + 2b^2$ may be written $2(a^2 + b^2)$.

Ex. 1. The sum of $3ax$, $4ax$, and $9ax$, is $16ax$; and of $-2xz - 4xz - 7xz$ is $-13xz$.

Ex. 2. The sum of $4ax - 5ax - 2ax + 11ax$ is $8ax$; and of $3by - 7by + 4by - 6by$ is $-6by$.

Ex. 3. The sum of $2a + b - 3x$, $3a - 2b + x$, $a + b - 5x$ is $6a - 7x$.

Ex. 4. The sum of $5a - 10b + 3c$, $2b - 3a - 7c$, $-5c + 8b - 15a$ is $-13a - 9c$.

Ex. 5. The sum of $a^m - b^n + 3x^p$, $2a^m - 3b^n - x^p$, and $a^m + 4b^n - x^q$ is $4a^m + 2x^p - x^q$.

Ex. 6. The sum of $a - b - c$, $b + c - d$, $d - e + f$, $e - f - g$ is $a - g$.

Ex. 7. The sum of $a + b + c + d$, $a + b + c - d$, $a + b - c + d$, $a - b + c + d$, and $-a + b + c + d$ is $3(a + b + c + d)$.

Ex. 8. The sum of $5x - 3a + b + 7$, $2b - 3x - 4a - 9$, $3a - 5 + 2b$, $5b - c + a$, $3 + 6a - 2c$, $2c - 10 + 5a - 3b$, and $2a - 9 - 4c + 3b$ is $2x + 10a + 10b - 5c - 23$.

Ex. 9. The sum of $5xy - 7ez + 18ax - 14by$, $3xy - 5cd + 11eg + 14ez$, $13ax + 20eg - 35cd + 18$, and $25xy - 15eg + 9by - 12ax$ is $33xy + 7ez + 19ax - 5by - 40cd + 16eg + 18$.

Ex. 10. The sum of $x^4 + ax^3 - bx^2 + 2cx - d$, $3ax^4 + cx^3 - 4a$, $2cx^3 - dx^2 - 3ax + 2b$, $x^3 - 3x^2 + 3x - 1$, and $4 - 2cx + 3bx^2 - 4ax^3 + x^4$ is $(3a+2)x^4 + (2c + 1 - 3a)x^3 + (2b + c - d - 3)x^2 + 3(1 - a)x - 4a + 2b - d + 3$.

SUBTRACTION.

Art. 16. The difference of two quantities in Algebra is expressed by simply combining the sign $-$ with that of the quantity to be subtracted. Thus, the difference of a and b is $a - b$; of a and $-b$ is $a - (-b)$ or $a + b$; of $-e$ and f is $-e - f$, and of $-g$ and $-h$, is $-g + h$. See below.

To subtract quantities. Change the signs of the quantities to be subtracted, or suppose them changed, and then add them to the other quantities agreeably to the rules of Addition.

(1)	$10a$	(2)	$5a - 12b$	(3)	$6x - 8y + 3$
	$7a$		$2a - 5b$		$2x + 9y - 2$
	$3a$		$3a - 7b$		$4x - 17y + 5$

The reason of the rule for Subtraction may be explained thus. Let it be required to subtract $2p - 3q$ from $m + n$. Now if we subtract $2p$ from $m + n$, there will remain $m + n - 2p$; but if we are to subtract $2p - 3q$ which is $< 2p$ from $m + n$, it is evident that the remainder will be greater by a quantity $= 3q$, *i. e.* the remainder will be $m + n - 2p + 3q$; or $m + n - (2p - 3q) = m + n - 2p + 3q$. Similarly, $a - (-b) = a + b$, $a - (b - c + d) = a - b + c - d$; and therefore if the sign $-$ be prefixed to any number of quantities included between brackets, when the brackets are taken away, the sign of every term must be changed; but if the sign $+$ be prefixed, the terms will be unaltered.

Ex. 1. $2x + 3y - (x - 2y) = x + 5y$.

Ex. 2. $a + b - (a - b) = 2b$ and $a - b - (a + b) = -2b$.

Ex. 3. $6a^2 + 12ab + 19b^2 + c^2 - (4a^2 + 8ab + 13b^2) = 2a^2 + 4ab + 6b^2 + c^2$.

Ex. 4. $5xy - 2 + 8x - y - (3xy - 8 - 8x - 3y) = 2xy + 6 + 16x + 2y$.

Ex. 5. $a^2 - ax - y^2 - (b^2 - by + u^2) = a^2 - ax - y^2 - b^2 + by - u^2$.

Ex. 6. Subtract $a^2xy + 3bx^2y - 13cxy^2 + 20y^5$ from $8a^2xy - 5bx^2y + 17cxy^2 - 9y^5$. Ans. $7a^2xy - 8bx^2y + 30cxy^2 - 29y^5$.

Ex. 7. Subtract $3(a+b) - 4(c+d)$ from $4(a+b) - 3(c+d)$. Ans. $a+b+c+d$.

Ex. 8. Subtract $y^3 - xy^2 + 3x - r$ from $y^3 - ay^2 + 2y - d$. Ans. $(x-a)y^2 + 2y - 3x + r - d$.

Ex. 9. Reduce to its simplest form $a - (b-c) + b - (a-c) + c - (a-b)$. Ans. $-a + b + 3c$.

Ex. 10. Find the value of $3(x^2+y^2) - \{(x^2 + 2xy + y^2) - (2xy - x^2 - y^2)\}$. Ans. $x^2 + y^2$.

Obs. When, as in Ex. 10, one or more brackets are included within another, it is better to take them away separately. Thus, $a - \{b + c - (x+y) + (d-e) - x(1+a)\}, = a - b - c + (x+y) - (d-e) + x(1+a) = a - b - c + x + y - d + e + x + ax$.

Also, it is evident that since a $-$ sign before a bracket changes all the signs of a bracket, if we incorporate any number of terms into a bracket with a $-$ sign before it, we must first change all the signs of the terms to be enclosed. Thus,

$$a + b - c + d - e + f + g - h = a + b - (c - d + e - f - g + h).$$

MULTIPLICATION.

Art. 17. When two simple algebraical quantities are multiplied together, the product has a *positive* or *negative* sign, according as the signs of the factors are the *same* or *different*.

To find the product of two or more *simple* quantities, multiply together the numeral coefficients, and set down all the letters, as in one word, prefixing the proper sign to the product. Thus,

$$a \times c = ac, \; 5b \times -4a = -20ab, \; -3ax \times 7b = -21abx,$$

$$-2ab \times -3cz = 6abcz, \; -2x \times 3y \times 4z = -24xyz.$$

Art. 18. When however the quantities to be multiplied are powers of the same root, the multiplication is effected by adding their exponents. Thus,

$$a \times a^3 = a^4,\ x^3 \times x^7 = x^{10} \text{ and } a^m \times a^n = a^{m+n}.$$

For $a \times a^3$ is identical with $a \times aaa$, or a^4.

So, $a^m \times a^n = aaa \dots$ to m factors $\times$ $aaa \dots$ to n factors

$= aaa \dots$ to $(m+n)$ factors $= a^{m+n}$.

Hence $4a \times bx \times -ab^2y = -4a^2b^3xy$.

Obs. When more than two quantities are multiplied together, the sign of the product is found by first finding the sign of the product of two of them, and then the sign of this product and the next quantity, and so on. Thus, $4a \times bx = 4abx$, and hence $4a \times bx \times -ab^2y = 4abx \times -ab^2y = -4a^2b^3xy$.

Art. 19. To find the product of *compound quantities*, multiply every term of the multiplicand by all the terms of the multiplier, by the preceding rule, and the sum of all these products will be the required product.

Ex. (1)
$$\begin{array}{l} 4a - 2b + c \\ 3a \\ \hline 12a^2 - 6ab + 3ac. \end{array}$$

Ex. (2)
$$\begin{array}{rr} 2x + y & \\ x - 2y & \\ \hline 2x^2 + xy & \\ & -4xy - 2y^2 \\ \hline 2x^2 & -3xy - 2y^2. \end{array}$$

In Ex. (2) $2x^2 + xy$ is the product by x, and $-4xy - 2y^2$ the product by $-2y$. It is usual to begin to multiply by the symbol on the left-hand, and to place *like* quantities under each other.

Ex. (3)
$$\begin{array}{lllll} a & -b & +c & & \\ a & +b & -c & & \\ \hline a^2 & -ab & +ac & & \\ & ab & -b^2 & +bc & \\ & & -ac & +bc & -c^2 \\ \hline a^2 & & -b^2 & +2bc & -c^2. \end{array}$$

Ex. (4)
$$\begin{array}{llll} a^2 & -ab & +b^2 & \\ a & +b & & \\ \hline a^3 & -a^2b & +ab^2 & \\ & a^2b & -ab^2 & +b^3 \\ \hline a^3 & & & +b^3 \end{array}$$

In Ex. 4. the arrangement of the terms has been made according to dimensions of a, in order that the like quantities might be found in the same vertical rows.

Ex. 5. Find the product of the factors $x + a$, $x + b$, $x + c$.

$$\begin{array}{l}
x + a \\
\underline{x + b} \\
x^2 + ax \\
\underline{\quad\; bx + ab} \\
x^2 + (a + b)\,x + ab, \text{ the product of } x + a \text{ and } x + b. \\
\underline{x \;+ c} \\
x^3 + (a + b)\,x^2 + abx \\
\underline{\qquad\quad cx^2 + (ac + bc)\,x + abc} \\
x^3 + (a + b + c)\,x^2 + (ab + ac + bc)\,x + abc, \text{ the required product.}
\end{array}$$

In this Example, where two or more terms in the product have a common factor, as x^2, x, we abbreviate the expression by means of a bracket.

Ex. 6. Multiply $x^2 + ax + b$ by $x^2 - ax + c$.

$$\begin{array}{l}
x^2 + ax + b \\
\underline{x^2 - ax + c} \\
x^4 + ax^3 + bx^2 \\
\quad - ax^3 - a^2x^2 - abx \\
\underline{\qquad\qquad\quad cx^2 + acx + bc} \\
x^4 + (b + c - a^2)\,x^2 - (b - c)\,ax + bc.
\end{array}$$

Art. 20. To prove the *Rule of signs*, let us suppose it required to find the product of $a - b$ by $c - d$.

Now since multiplication is a repeated addition of the multiplicand as often as the multiplier contains unity, therefore $a - b$ is to be taken as often as there are units in $c - d$. Now if $a - b$ be taken c times, the result will evidently be too great by the quantity $a - b$ taken d times. But from the nature of addition $a - b$ taken c times $= ca - cb$, and $a - b$ taken d times $= da - db$, therefore the product required will be $da - db$ subtracted from $ca - cb$. And this by subtraction $= ca - cb - da + db$, and therefore the product arising from the multiplication of $a - b$ by $c - d$ is $ca - cb - da + db$, or $(a - b)(c - d) = ac - bc - ad + bd$.

Now let $b = 0$, and $d = 0$, $\therefore\ (+a) \times (+c) = +ac$

$b = 0$, and $c = 0$, $\therefore\ (+a) \times (-d) = -ad$

$a = 0$, and $d = 0$, $\therefore\ (-b) \times (+c) = -bc$

$a = 0$, and $c = 0$, $\therefore\ (-b) \times (-d) = +bd$;

or, when the signs are the *same*, the product is positive, when *different*, negative. The same is also evident by examining the product, and considering how each of its terms originates.

ART. 21. Ex. 1. Multiply $x+2$ by $x+3$. Ans. x^2+5x+6.

Ex. 2. Multiply x^2+ax+a^2 by $x+a$, and $x-a$ by $x-b$.
Ans. $x^3+2ax^2+2a^2x+a^3$, and $x^2-(a+b)x+ab$.

Ex. 3. Multiply a^2+ax+x^2 by a^2-ax+x^2.
Ans. $a^4+a^2x^2+x^4$.

Ex. 4. Multiply a^2+ab+b^2 by $a-b$. Ans. a^3-b^3.

Ex. 5. Multiply $2a+3y-5x$ by $2a+3x$.
Ans. $4a^2+6ay-4ax+9xy-15x^2$.

Ex. 6. Multiply $16x^4-8x^3y+4x^2y^2-2xy^3+y^4$ by $2x+y$.
Ans. $32x^5+y^5$.

Ex. 7. Prove that $(1-x+x^2-x^3)(1+x)=1-x^4$.

Ex. 8. Prove that $(1+x+x^4+x^5)(1-x+x^2-x^3)=1-x^8$.

Ex. 9. Multiply $6x^2-4xy-y^2$ by $12x^2-5xy-3y^2$.
Ans. $72x^4-78x^3y-10x^2y^2+17xy^3+3y^4$.

Ex. 10. Multiply $7x-2y-9$ by $3x-11y$.
Ans. $21x^2-83xy-27x+22y^2+99y$.

Ex. 11. Multiply $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.
Ans. $a^3+b^3+c^3-3abc$.

Ex. 12. Multiply $m+n+p+q$ by $m-n-p-q$.
Ans. $m^2-n^2-p^2-q^2-2np-2nq-2pq$.

Ex. 13. Multiply $5x^2-3xy+7y^2$ by $3x-y$.
Ans. $15x^3-14x^2y+24xy^2-7y^3$.

Ex. 14. Prove that $(x+2)(x+4)(x+6)=x^3+12x^2+4x+48$.

Ex. 15. Prove that $(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

Ex. 16. Prove that $(x-3)(x+3)(x-4)(x+4)=x^4$ $25x^2+144$.

Ex. 17. Prove that $(a^3+3a^2x+3ax^2+x^3)(a^3-3a^2x+3ax^2$ $x^3)=a^6-3a^4x^2+3a^2x^4-x^6$.

Ex. 18. Prove that $(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4$.

Ex. 19. Prove that $x(x+1)(x+2)+x(x-1)(x-2)+4(x-1)x(x+1)=6x^3$.

Ex. 20. Find the coefficient of x^9 in the product of

$$x^{13}-ax^{12}+bx^{11}-cx^{10}+ax^9-bx^7+ax^6+bx^3 \text{ by}$$
$$x^{21}-ax^{17}+\beta x^{11}-\gamma x^6+\delta x^5+\epsilon x^4-ax^3+\beta x^2-\delta x-\epsilon.$$

Ans. $-b(\beta+\gamma)$.

DIVISION.

Art. 22. Since Division is the reverse of Multiplication, the rules for its operation will be deduced from reversing the operation of Multiplication.

If the divisor and dividend have *like* signs, the quotient will be positive, if *unlike*, negative. This is evident from the consideration that the quotient must be such a quantity, that when multiplied by the divisor it must give the dividend with its proper sign. Thus,

$$\because\ (+a)\times(+b)=+ab,\ \therefore\ \frac{+ab}{+a}=+b;$$
$$\because\ (-a)\times(+b)=-ab,\ \therefore\ \frac{-ab}{-a}=+b;$$
$$\because\ (+a)\times(-b)=-ab,\ \therefore\ \frac{-ab}{+a}=-b;$$
$$\text{and}\ \because\ (-a)\times(-b)=+ab,\ \therefore\ \frac{+ab}{-a}=-b.$$

Art. 23. When the divisor is *simple*, and a factor of every term of the dividend, the quotient is found by dividing each term of the dividend by the coefficient of the divisor, and cancelling out of each term of the dividend, the letter or letters common. Thus, $\dfrac{12abc}{3ac}=4b$, $\dfrac{16a^3xy-28a^2x^2+4a^2x^3}{4a^2x}=4ay-7x+x^2$. These results can easily be verified by multiplication. When however the quantities to be divided are powers of the same root, the division is effected by subtracting their exponents.

Thus, $\frac{a^5}{a^3} = a^2$, $\frac{a^m}{a^n} = a^{m-n}$.

For $\frac{a^5}{a^3} = \frac{aaaaa}{aaa} = aa = a^2$;

So $\frac{a^m}{a^n} = \frac{aa \ldots \text{ to } m \text{ factors}}{aa \ldots \text{ to } n \text{ factors}} = aa \ldots \text{ to } (m-n) \text{ factors} = a^{m-n}$.

Obs. If m be $< n$, then $\frac{a^m}{a^n} = a^{m-n} = a^{-(n-m)} = \frac{1}{a^{n-m}}$.

Thus, $\frac{a^3}{a^5} = \frac{1}{a^2}$.

ART. 24. If the divisor be a factor of part of the dividend only, then the quotient is represented as a mixed number. Thus,

$$\frac{x^3 + ax^2 + ax}{x^2} = x + a + \frac{a}{x}.$$

Also, if the divisor be not a factor of any part of the dividend, the quotient is expressed by a fraction whose numerator is the dividend, and denominator the divisor. Thus, $3ad^2$ divided by $2mbc = \frac{3ad^2}{2mbc}$.

ART. 25. When the divisor is *compound*, arrange both the divisor and dividend according to the dimensions of some letter common to both; divide the first term of the dividend by the first term of the divisor, observing the rule of signs: multiply this quotient by the whole divisor, and subtract the product from the dividend.

Consider the remainder, if there be any, as the new dividend, and proceed as before, joining this quotient to the former one, and so on.

Ex. 1. Divide $8a^2 + 2ab - 15b^2$ by $2a + 3b$.

$$\begin{array}{r|l} 2a+3b)\ 8a^2 + 2ab - 15b^2 & (4a - 5b \\ \underline{8a^2 + 12ab } & \\ -10ab - 15b^2 & \\ \underline{-10ab - 15b^2} & \end{array}$$

so that the quotient is $4a - 5b$; the terms being arranged according to the dimensions of the letter a.

In this Example, we first divide $8a^2$ by $2a$ and get the quotient $4a$; then we multiply the whole divisor by it, and subtract the product from the dividend. Then, dividing the first term of the remainder $-10ab$, by $2a$, we get $-5b$, and no remainder. The result may be verified by multiplying the divisor by the quotient.

Ex. 2. Divide $3a^3 - 12a^2 - a^2b + 10ab - 2b^2$ by $3a - b$.

$$
\begin{array}{l}
3a - b)\ 3a^3 - 12a^2 - a^2b + 10ab - 2b^2\ (a^2 - 4a + 2b \\
\qquad\quad \underline{3a^3 - a^2b} \\
\qquad\qquad -12a^2 + 10ab \\
\qquad\qquad \underline{-12a^2 + \ \ 4ab} \\
\qquad\qquad\qquad\qquad 6ab - 2b^2 \\
\qquad\qquad\qquad\qquad \underline{6ab - 2b^2}
\end{array}
$$

Ex. 3. Divide $a^3 + b^3$ by $a + b$, and $a^6 - b^6$ by $a^3 - b^3$.

$$
\begin{array}{l}
a + b)\ a^3 + b^3\ (a^2 - ab + b^2, \\
\qquad\ \ \underline{a^3 + a^2b} \\
\qquad\quad -a^2b + b^3 \\
\qquad\quad \underline{-a^2b - ab^2} \\
\qquad\qquad\quad ab^2 + b^3 \\
\qquad\qquad\quad \underline{ab^2 + b^3}
\end{array}
\qquad
\begin{array}{l}
a^3 - b^3)\ a^6 - b^6\ (a^3 + b^3 \\
\qquad\qquad \underline{a^6 - a^3b^3} \\
\qquad\qquad\quad a^3b^3 - b^6 \\
\qquad\qquad\quad \underline{a^3b^3 - b^6}
\end{array}
$$

Ex. 4. Divide $x^8 + 16x^4y^4 + 256y^8$ by $x^4 - 4x^2y^2 + 16y^4$.

$$
\begin{array}{l}
x^4 - 4x^2y^2 + 16y^4)\ x^8 + 16x^4y^4 + 256y^8\ (x^4 + 4x^2y^2 + 16y^4 \\
\qquad\qquad\qquad\quad \underline{x^8 - 4x^6y^2 + 16x^4y^4} \\
\qquad\qquad\qquad\qquad\quad 4x^6y^2 + 256y^8 \\
\qquad\qquad\qquad\qquad\quad \underline{4x^6y^2 - 16x^4y^4 + 64x^2y^6} \\
\qquad\qquad\qquad\qquad\qquad\qquad 16x^4y^4 - 64x^2y^6 + 256y^8 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{16x^4y^4 - 64x^2y^6 + 256y^8}.
\end{array}
$$

In this Example, each remainder is arranged according to the highest powers of x.

Ex. 5. Divide $mpx^3 + (mq - np)\,x^2 - (mr + nq)\,x + nr$ by $mx - n$.

$$
\begin{array}{l}
mx - n)\ mpx^3 + (mq - np)\,x^2 - (mr + nq)\,x + nr\ (px^2 + qx - r \\
\qquad\qquad \underline{mpx^3 \ - \ npx^2} \\
\qquad\qquad\qquad mqx^2 - (mr + nq)\,x \\
\qquad\qquad\qquad \underline{mqx^2 - nqx} \\
\qquad\qquad\qquad\qquad -mrx + nr \\
\qquad\qquad\qquad\qquad \underline{-mrx + nr}.
\end{array}
$$

Ex. 6. Divide $x^3 - px^2 + qx - r$ by $x - a$.

$$
\begin{array}{l|l|l}
x - a & x^3 - px^2 + qx - r & x^2 + (a-p)\,x + (a^2 - ap + q) \\
 & x^3 - ax^2 & \\
\hline
 & (a-p)\,x^2 + qx & \\
 & (a-p)\,x^2 - (a^2 - ap)\,x & \\
\hline
 & (a^2 - ap + q)\,x - r & \\
 & (a^2 - ap + q)\,x - (a^3 - a^2p + aq) & \\
\hline
 & a^3 - a^2p + aq - r. &
\end{array}
$$

In this Example, there is a remainder $a^3 - a^2p + aq - r$. This is sometimes written as a fraction, of which it is the numerator, and the divisor the denominator.

Ex. 7. Divide 1 by $1 - x$.

$$
\begin{array}{l|l|l}
1 - x & 1 & 1 + x + x^2 + x^3 + \ldots \\
 & 1 - x & \\
\hline
 & x & \\
 & x - x^2 & \\
\hline
 & x^2 & \\
 & x^2 - x^3 & \\
\hline
 & x^3 & \\
 & x^3 - x^4 & \\
\hline
 & x^4 &
\end{array}
$$

In this Example, the quotient will never terminate; it may be completed, as in Arithmetic, by annexing to it a fraction, whose numerator is the remainder, and denominator the divisor.

Thus $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dfrac{x^4}{1-x}$; and this result with its remainder, may, by fractions, be restored to the form $\dfrac{1}{1-x}$. (Vid. Art. 30.)

If there be any quantity which is common to every term of the divisor and dividend, it will be simpler first to divide by that quantity, and then by the other part of the divisor, as in

Ex. 8. Divide $x^5 - a^4x$ by $x^2 - ax$, or $(x - a)\,x$.

First dividing by x, we have $x^4 - a^4$, which remains to be divided by $x - a$, and the quotient is $x^3 + ax^2 + a^2x + a^3$.

In the operation of Division, all the terms of the quotient are multiplied by all the terms of the divisor, and the product subtracted from the dividend. This therefore must give the true quotient.

ART. 26. Ex. 1. Divide $a^2 - 2ab + b^2$ by $a - b$. Ans. $a - b$.

Ex. 2. Divide $9x^3 - 72$ by $3x - 6$. Ans. $3x^2 + 6x + 12$.

Ex. 3. Divide $y^4 - 81$ by $y - 3$. Ans. $y^3 + 3y^2 + 9y + 27$.

Ex. 4. Divide $2a^4 - 32$ by $a - 2$. Ans. $2a^3 + 4a^2 + 8a + 16$.

Ex. 5. Divide $a^4 + a^2x^2 + x^4$ by $a^2 + ax + x^2$. Ans. $a^2 - ax + x^2$.

Ex. 6. Divide $x^5 - 3x^4 + 8x^3 - 16x^2 + 23x - 21$ by $x^2 - 2x + 3$. Ans. $x^3 - x^2 + 3x - 7$.

Ex. 7. Divide $x^3 - 86x - 140$ by $x - 10$. Ans. $x^2 + 10x + 14$.

Ex. 8. Divide $6x^4 - x^3y - 8x^2 - 2x^2y^2 + 17xy - 30$ by $3x^2 - 2xy + 5$. Ans. $2x^2 + xy - 6$.

Ex. 9. Divide $(x^2 - 4)(x^2 - 4x)$ by $x^2 + 2x$. Ans. $x^2 - 6x + 8$.

Ex. 10. Divide $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$ by $a^4 + a^3b + a^2b^2 + ab^3 + b^4$. Ans. $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

Ex. 11. Divide $1 - 5y + 10y^2 - 10y^3 + 5y^4 - y^5$ by $1 - 3y + 3y^2 - y^3$. Ans. $1 - 2y + y^2$.

Ex. 12. Divide $x^4 - 4x^3 + 6x^2 - 4x + 1$ by $x^2 - 2x + 1$. Ans. $x^2 - 2x + 1$.

Ex. 13. Divide $x^8 + 4a^2x^4 + 16a^4$ by $x^4 - 2ax^2 + 4a^2$. Ans. $x^4 + 2ax^2 + 4a^2$.

Ex. 14. Divide $x^3 - apx^2 + a^2px - a^3$ by $x - a$. Ans. $x^2 - a(p - 1)x + a^2$.

Ex. 15. Divide $x^3 + px^2 + qx + r$ by $x + a$.

Ans. $x^2 - (a - p)x + a^2 - ap + q - \dfrac{a^3 - a^2p + aq - r}{x + a}$.

Ex. 16. Divide $a^{m+n} - a^m b^n + a^n b^m - b^{m+n}$ by $a^n - b^n$. Ans. $a^m + b^m$.

Ex. 17. Divide $-x^4y^2 + y^2z^4 - x^2z^4 - x^6 - 2x^4z^2 + y^6 + 2y^4z^2 + x^2y^4$ by $-x^2 - y^2 - z^2$. Ans. $x^4 + x^2z^2 - y^2z^2 - y^4$.

Ex. 18. Divide $a^2 + (a - 1)x^2 + (a - 1)x^3 + (a - 1)x^4 - x^5$ by $a - x$. Ans. $a + x + x^2 + x^3 + x^4$.

Ex. 19. Divide $1 + 2x$ by $1 - 3x$. Ans. $1 + 5x + 15x^2 + 45x^3 + \ldots$.

Ex. 20. Divide $1 + 2x$ by $1 - x - x^2$. Ans. $1 + 3x + 4x^2 + 7x^3 + 11x^4 + \ldots$.

Ex. 21. Divide a by $1-x$. Ans. $a+ax+ax^2+\ldots.$

Ex. 22. Divide x^5+1 by $x+1$. Ans. $x^4-x^3+x^2-x+1$.

Ex. 23. Divide $x^4-px^3+9x^2-rx$ by x^2-ax.

$$\text{Ans.}\quad x^2+(a-p)\,x+a^2-pa+9+\frac{a^3-pa^2+9a-r}{x-a}.$$

Ex. 24. Divide $a^5-4a^4b+10a^3b^2-8a^2b^3+ab^4+12b^5$ by $a^2-2ab+3b^2$. Ans. $a^3-2a^2b+3ab^2+4b^3$.

ART. 27. All the fundamental operations in Arithmetic can be performed in a similar manner to those in Algebra.

Add and subtract 216 and 348.

$$\begin{array}{rl} & 300+40+8 \\ & \underline{200+10+6} \\ \text{sum} = & \underline{500+50+14} = 564 \end{array} \qquad \begin{array}{rl} & 300+40+8 \\ & \underline{200+10+6} \\ \text{difference} & \underline{100+30+2} = 132 \end{array}$$

Multiply 216 by 25.

$$\begin{array}{rrrrl} & 200 & +\ 10 & +\ 6 & \\ & & 20 & +\ 5 & \\ \hline 4000 & +\ 200 & +\ 120 & & \\ & +\ 1000 & +\ 50 & +\ 30 & \\ \hline 4000 & +\ 1200 & +\ 170 & +\ 30 & = 5400. \end{array}$$

Divide 5400 by 25.

$$\begin{array}{l} 20+5)\ 4000+1200+170+30\ (200+10+6=216. \\ \underline{4000+1000} \\ 200+170 \\ \underline{200+\ \ 50} \\ 120+30 \\ \underline{120+30} \end{array}$$

Similarly, $35^2=(30+5)^2=(30+5)\,(30+5)$.

Extract the square root of 169.

$$\begin{array}{rl} & 169\ (10+3=13 \\ 10^2 = & \underline{100} \\ 2\times10+3=23) & \ \ 69 \\ & \ \ \underline{69} \end{array}$$

and similarly for cube root. Vide Arts. 57 and 58.

CHAPTER III.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF SIMPLE ALGEBRAICAL FRACTIONS.

ART. 28. THE nature of fractions in Algebra is precisely the same as in Arithmetic, and the operations performed on them are the same as those performed on numerical fractions. (Vid. Arithmetic, Art. 18.)

$\frac{a}{a+x}$ is a *proper* fraction, the numerator being $<$ than denominator.

$\frac{a}{a}$ or $\frac{a+x}{a}$ are *improper* fractions, the numerator being not $<$ than denominator.

$b+\frac{x}{a}$ is a *mixed* quantity.

$\frac{a}{b}$ of $\frac{c}{d}$ is a *compound* fraction.

$\dfrac{a+\frac{b}{c}}{x+\frac{y}{z}}$ is a *complex* fraction.

The fraction $\frac{b}{a}$ is called the *reciprocal* of $\frac{a}{b}$, as c is called the reciprocal of $\frac{1}{c}$, or $\frac{1}{c}$ of c.

ART. 29. If both the numerator and denominator of a fraction be multiplied or divided by the same quantity, the value of the fraction will not be altered.

Let any fraction, as $\frac{a}{b}=x$, then $\frac{a}{b}\times b=bx$, or $a\times\frac{b}{b}=bx$, or $a=bx$.

$$\therefore\ na = nbx, \text{ multiplying by } n,$$

$$\therefore \text{ dividing by } nb,\ \frac{na}{nb} = x = \frac{a}{b}; \text{ or } \frac{na}{nb} = \frac{a}{b}.$$

Hence the numerator and denominator of the fraction $\frac{a}{b}$ may be multiplied by n, and its value will remain unaltered; or the numerator and denominator of the fraction $\frac{na}{nb}$ be divided by n, and its value will remain the same.

$$\text{Similarly, } \frac{ax}{ay} = \frac{x}{y},\quad \frac{a^3 + a^2x}{a^4} = \frac{a+x}{a^2}$$

By multiplying the numerator and denominator of the fraction $\frac{a}{b}$ by -1, we have $\frac{a}{b} = \frac{-a}{-b}$, or the signs of both the numerator and denominator of a fraction may be changed without altering the value of the fraction.

$$\text{Thus, } \frac{x^2yz}{-axy} = -\frac{xyz}{a},\quad -\frac{2}{x(4x^2-1)} = \frac{2}{x(1-4x^2)}.$$

Art. 30. An integral quantity is represented as a fraction by placing unity below it.

$$\text{Thus, } a = \frac{a}{1},\quad a + b + \frac{c}{y} = \frac{a+b}{1} + \frac{c}{y}.$$

A mixed quantity is represented as an improper fraction by multiplying the integral part by the denominator of the fraction, adding the numerator to the product, and placing the denominator under their sum.

$$\text{Thus, } x + \frac{x^2}{a} = \frac{ax + x^2}{a},\quad x - \frac{a^2 - x^2}{x} = \frac{x^2 - a^2 + x^2}{x} = \frac{2x^2 - a^2}{x};$$

$$a - x + \frac{x^2}{a+x} = \frac{(a-x)(a+x) + x^2}{a+x} = \frac{a^2 - x^2 + x^2}{a+x} = \frac{a^2}{a+x}.$$

An improper fraction is expressed as a mixed quantity, by dividing the numerator by the denominator for the integral part, and placing the remainder, if any, over the denominator.

$$\text{Thus, } \frac{ax + x^2}{a} = x + \frac{x^2}{a},\quad \frac{ax + 2x^2}{a+x} = x + \frac{x^2}{a+x},\quad \frac{x^2 - y^2}{x+y} = x - y.$$

Obs. If the sign $-$ precedes a fraction whose numerator is compound, it changes all the signs of the terms of the numerator.

$$\text{Thus, } a - \frac{x-y-z+l-m}{c+d} = \frac{a(c+d)-x+y+z-l+m}{c+d}.$$

ART. 31. A fraction is reduced to its lowest terms, or simplest form, by continually dividing the numerator and denominator by any common factor, until they contain no common factor.

$$\text{Thus, } \frac{a^2xy^2}{4a^2x^2y} = \frac{y}{4x}, \text{ dividing by the factor } a^2xy$$

$$\frac{ax-x^2}{a^2-x^2} = \frac{(a-x)x}{a^2-x^2} = \frac{x}{a+x}, \text{ dividing by } a-x,$$

$$\frac{a^2-2ax+x^2}{a^2-x^2} = \frac{(a-x)^2}{a^2-x^2} = \frac{(a-x)^2}{(a+x)(a-x)} = \frac{a-x}{a+x},$$

$$\frac{a^4-b^4}{a^5-a^3b^2} = \frac{(a^2+b^2)(a^2-b^2)}{a^3(a^2-b^2)} = \frac{a^2+b^2}{a^3},$$

$$\frac{56a^2bc}{24adc^2} = \frac{7ab}{3dc}, \text{ dividing by } 8ac.$$

It is evident that cases may occur, where it is impossible to discover by inspection, the common factor of the numerator and denominator, and then we must have recourse to the G. C. M. of the quantities. But the above method by inspection, will be found sufficient in the Examples generally given in the fundamental operations of Fractions.

Obs. It will greatly facilitate the splitting of the numerator or denominator into factors, to remember that, the product of the sum and difference of any two quantities is always equal to the difference of their squares.

Thus, $x^2-y^2=(x+y)(x-y)$, $a^4-b^4=(a^2+b^2)(a^2-b^2)$.

(*Euclid*, II. 5.)

Also, $(x+y)^2=x^2+2xy+y^2$ (*Euc.* II. 4.), $(x-y)^2=x^2-2xy+y^2$.

Ex. 1. Find the value of $\dfrac{x^6-y^6}{x^3-y^3}$. Ans. x^3+y^3.

Ex. 2. Find the value of $\frac{a^2x - x^3}{a^3 - 2a^2x + ax^2}$. Ans. $\frac{ax + x^2}{a^2 - ax}$.

Ex. 3. Find the value of $\frac{a^3 + a^2x}{ax + a^4}$. Ans. $\frac{a^2 + ax}{x + a^3}$.

ART. 32. To reduce fractions to others having a common denominator.

Multiply each numerator separately into all the denominators, except its own, for a new numerator, and all the denominators together for a common denominator.

Reduce $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, to a common denominator.

$$\frac{a}{b} = \frac{a \times d \times f}{b \times d \times f} = \frac{adf}{bdf}.$$

$$\frac{c}{d} = \frac{c \times b \times f}{d \times b \times f} = \frac{cbf}{bdf}.$$

$$\frac{e}{f} = \frac{e \times b \times d}{f \times b \times d} = \frac{ebd}{bdf}.$$

Similarly,

$\frac{ax}{a - x}$ and $\frac{a^2 - x^2}{a + x}$ become $\frac{a^2x + ax^2}{a^2 - x^2}$ and $\frac{a^3 - a^2x - ax^2 + x^3}{a^2 - x^2}$;

and $\frac{a + 1}{a - 1}$, $\frac{a - 1}{a + 1}$ and $\frac{a^2 - 1}{a^2 + 1}$ become $\frac{a^4 + 2a^3 + 2a^2 + 2a + 1}{a^4 - 1}$,

$\frac{a^4 - 2a^3 + 2a^2 - 2a + 1}{a^4 - 1}$, and $\frac{a^4 - 2a^2 + 1}{a^4 - 1}$ respectively.

If however the denominators of the proposed fractions have a common measure, the fractions may be reduced to others having a lower common denominator than that found from above; thus, if the fractions be $\frac{ab}{cd}$ and $\frac{fg}{de}$, we have

$$\frac{ab}{cd} = \frac{abe}{cde}, \quad \frac{fg}{de} = \frac{cfg}{cde}.$$

Hence the least common denominator is the L. C. M. of the proposed denominators, and the numerators are obtained by multiplying the original numerators by the quotients arising from the division of the L. C. M. by the corresponding denominators. In the Examples in the fundamental operations of Fractions, the L. C. M. can be determined by inspection.

Reduce $\frac{a}{x-1}$, $\frac{b}{x^2-1}$, and $\frac{c}{x^2-2x+1}$, to their least common denominator.

The least common measure of $x-1$, x^2-1, or $(x-1)(x+1)$, and x^2-2x+1, or $(x-1)^2$ is $(x-1)^2(x+1)$ or x^3-x^2-x+1, and the fractions are

$$\frac{a}{x-1} = \frac{a(x-1)(x+1)}{x^3-x^2-x+1} = \frac{a(x^2-1)}{x^3-x^2-x+1},$$

$$\frac{b}{x^2-1} = \frac{b(x-1)}{x^3-x^2-x+1},$$

$$\frac{c}{x^2-2x+1} = \frac{c(x+1)}{x^3-x^2-x+1}.$$

Similarly, the fractions $\frac{x^2}{a^2+ax}$, $\frac{a^2}{ax-x^2}$, and $\frac{ax}{a^2-x^2}$, become $\frac{ax^3-x^4}{ax(a^2-x^2)}$, $\frac{a^4+a^3x}{ax(a^2-x^2)}$, and $\frac{a^2x^2}{ax(a^2-x^2)}$, respectively.

Obs. If the denominators of the proposed fractions be numeral, then their L. C. M. is found by Arithmetic.

ADDITION.

ART. 33. To find the sum of two fractions.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions, and let $\frac{a}{b} = x$, $\frac{c}{d} = y$.

$$\therefore\ a = bx,\ c = dy;$$

$$\therefore\ ad = bdx,\ bc = bdy;$$

$$\therefore\ ad + bc = bdx + bdy = bd(x+y);$$

$$\therefore\ x + y \text{ or } \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Hence the following rule. Reduce them to a common denominator, add the numerators, and place the sum over the common denominator.

Ex. 1. Add together $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$.

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf+bcf+bde}{bdf}. \quad \text{Art. 32.}$$

Ex. 2. Add together $\frac{ax}{a-x}$, and $\frac{a^2-x^2}{a+x}$.

$$\frac{ax}{a-x} + \frac{a^2-x^2}{a+x} = \frac{ax(a+x)}{a^2-x^2} + \frac{(a-x)(a^2-x^2)}{a^2-x^2}$$

$$= \frac{a^2x + ax^2 + a^3 - a^2x - ax^2 + x^3}{a^2-x^2}$$

$$= \frac{a^3+x^3}{a^2-x^2}.$$

Ex. 3. Add together

$$\frac{2}{x+a},\ \frac{3a}{(x+a)^2},\ \text{and}\ \frac{3a-2x}{x^2-2ax+3a^2}.$$

$$\frac{2}{x+a} + \frac{3a}{(x+a)^2} = \frac{2(x+a)+3a}{(x+a)^2} = \frac{2x+5a}{(x+a)^2};$$

$$\therefore\ \frac{2}{x+a} + \frac{3a}{(x+a)^2} + \frac{3a-2x}{x^2-2ax+3a^2} = \frac{2x+5a}{(x+a)^2} + \frac{3a-2x}{x^2-2ax+3a^2}$$

$$= \frac{(2x+5a)(x^2-2ax+3a^2)+(3a-2x)(x^2+2ax+a^2)}{(x+a)^2(x^2-2ax+3a^2)}$$

$= \dfrac{18a^3}{x^4+4a^3x+3a^4}$, by multiplying out the numerator and denominator.

Ex. 4. Add together

$$\frac{1}{4a^3(a+x)},\ \frac{1}{4a^3(a-x)},\ \text{and}\ \frac{1}{2a^2(a^2+x^2)}.$$

$$\frac{1}{4a^3(a+x)} + \frac{1}{4a^3(a-x)} = \frac{(a-x)+(a+x)}{4a^3(a^2-x^2)}$$

$$= \frac{2a}{4a^3(a^2-x^2)} = \frac{1}{2a^2(a^2-x^2)};$$

$$\therefore\ \text{required sum} = \frac{1}{2a^2(a^2-x^2)} + \frac{1}{2a^2(a^2+x^2)},$$

$$= \frac{(a^2+x^2)+(a^2-x^2)}{2a^2(a^4-x^4)} = \frac{1}{a^4-x^4}.$$

Ex. 5. $\dfrac{a}{a+x} + \dfrac{x}{a-x} = \dfrac{a^2+x^2}{a^2-x^2}.$

Ex. 6. $\dfrac{a+x}{a-x} + \dfrac{a-x}{a+x} = \dfrac{2(a^2+x^2)}{a^2-x^2}.$

Ex. 7. $\dfrac{a-b}{a+b} + \dfrac{ab}{a^2-b^2} = \dfrac{a^2-ab+b^2}{a^2-b^2}.$

Ex. 8. $\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x-y}{x+y}=\dfrac{2(x^2-xy+y^2)}{x^2-y^2}$.

Ex. 9. $\dfrac{x}{1-x}+\dfrac{x^2}{(1-x)^2}=\dfrac{x}{(1-x)^2}$.

Ex. 10. $\dfrac{b}{d}+\dfrac{ad-bc}{d(c+dx)}=\dfrac{a+bx}{c+dx}$.

Ex. 11. $\dfrac{a}{c}+\dfrac{(ad-bc)x}{c(c-dx)}=\dfrac{a-bx}{c-dx}$.

Ex. 12. $\dfrac{1}{3(1+x)}+\dfrac{2-x}{3(1-x+x^2)}=\dfrac{1}{1+x^3}$.

Ex. 13. $\dfrac{a^2}{x+a}+\dfrac{b^2-2ab}{x+b}+\dfrac{(a-b)b^2}{(x+b)^2}=\dfrac{(a-b)^2x^2}{(x+a)(x+b)^2}$.

Ex. 14. $1+x+\dfrac{x^2}{1-x}=\dfrac{1}{1-x}$.

Ex. 15. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}$

$$=\frac{1}{abc}.$$

Ex. 16. $2m+\dfrac{m}{a^m-1}+\dfrac{a^m m}{1-a^m}=m$.

Ex. 17. $\dfrac{x-3y}{x+y}+\dfrac{x+3y}{x-y}=\dfrac{2(x^2+3y^2)}{x^2-y^2}$.

Ex. 18. $\dfrac{a+2b}{a+b}+\dfrac{a-2b}{a-b}=\dfrac{2(a^2-2b^2)}{a^2-b^2}$.

Ex. 19. $\dfrac{3a+2x}{3a-2x}+\dfrac{3a-2x}{3a+2x}=\dfrac{18a^2+8x^2}{9a^2-4x^2}$.

Ex. 20. $\dfrac{a^2+ab+b^2}{a+b}+\dfrac{b^2}{a-b}=\dfrac{a(a^2+b^2)}{a^2-b^2}$.

Ex. 21. $\dfrac{2x+1}{3x}+\dfrac{4x+2}{5x}+\dfrac{1}{7}=\dfrac{169x+77}{105x}$.

Ex. 22. $\dfrac{2x}{3}+\dfrac{3x}{2}+\dfrac{2x+1}{4}=\dfrac{32x+3}{12}$.

Ex. 23. $\dfrac{x}{e^x+1}+\dfrac{x}{e^x-1}=\dfrac{2xe^x}{e^{2x}-1}$.

Ex. 24. $\dfrac{1}{1+x} + \dfrac{1}{1-x} + \dfrac{2}{1+x^2} = \dfrac{4}{1-x^4}$.

Ex. 25. $\dfrac{x+2}{3} + \dfrac{x}{4} + \dfrac{x-5}{2} = \dfrac{13x-22}{12}$.

Ex. 26. $1+x+x^2+x^3+\dfrac{x^4}{1-x} = \dfrac{1}{1-x}$.

Obs. The sum of mixed quantities is found by adding the integral parts by themselves, and considering the result as a fraction whose denominator is unity.

SUBTRACTION.

Art. 34. To find the difference of two fractions.

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be any two fractions, and let $\dfrac{a}{b}=x$, $\dfrac{c}{d}=y$;

$$\therefore\ a=bx,\ c=dy;$$

$$\therefore\ ad=bdx,\ bc=bdy;$$

$$\therefore\ ad-bc=bdx-bdy=bd(x-y);$$

$$\therefore\ x-y, \text{ or } \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

Hence the following Rule. Reduce them to a common denominator, subtract the numerators, and place the difference over the common denominator.

Ex. 1. Subtract $\dfrac{x-y}{x+y}$ from $\dfrac{x+y}{x-y}$.

$$\frac{x+y}{x-y}-\frac{x-y}{x+y}=\frac{(x+y)^2-(x-y)^2}{(x+y)(x-y)}$$

$$=\frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{x^2-y^2}=\frac{4xy}{x^2-y^2}.$$

Ex. 2. Subtract $\dfrac{a}{a+x}$ from $\dfrac{a+x}{a}$.

$$\frac{a+x}{a}-\frac{a}{a+x}=\frac{(a+x)^2-a^2}{a^2+ax}=\frac{2ax+x^2}{a^2+ax}.$$

Ex. 3. Find the value of $\dfrac{3a-4b}{7}-\dfrac{2a-b-c}{3}+\dfrac{15a-4c}{12}$.

$$\frac{3a-4b}{7} - \frac{2a-b-c}{3} = \frac{3(3a-4b)-7(2a-b-c)}{21}$$

$$= \frac{9a-12b-14a+7b+7c}{21} = \frac{7c-5a-5b}{21};$$

$$\therefore \text{ value required} = \frac{7c-5a-5b}{21} + \frac{15a-4c}{12}$$

$$= \frac{84c-60a-60b+315a-84c}{252}$$

$$= \frac{255a-60b}{252} = \frac{85a-20b}{84}.$$

Ex. 4. $\dfrac{a^2}{a-b} - a = \dfrac{ab}{a-b}$.

Ex. 5. $\dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a-b}{a+b} = \dfrac{2ab}{a^2-b^2}$.

Ex. 6. $\dfrac{1}{1-x} - \dfrac{1}{1+x} = \dfrac{2x}{1-x^2}$.

Ex. 7. $a - \dfrac{a^2}{a+b} = \dfrac{ab}{a+b}$.

Ex. 8. $2 + \dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a-b}{a+b} = \dfrac{2(a^2+ab-b^2)}{a^2-b^2}$.

Ex. 9. $\dfrac{1}{1-x} - \dfrac{2}{1-x^2} = \dfrac{x-1}{1-x^2}$.

Ex. 10. $\dfrac{x+y}{y} - \dfrac{2x}{x+y} + \dfrac{x^3-x^2y}{y^3-x^2y} = \dfrac{y}{x+y}$.

Ex. 11. $\dfrac{a}{(a-b)(x+a)} - \dfrac{b}{(a-b)(x+b)} = \dfrac{x}{(x+a)(x+b)}$.

Ex. 12. $1-x+x^2 - \dfrac{x^3}{1+x} = \dfrac{1}{1+x}$.

Ex. 13. $\dfrac{x}{1-x} - \dfrac{x^2}{(1-x)^2} + \dfrac{x^3}{(1-x)^3} = x + \dfrac{x^4}{(1-x)^3}$.

Ex. 14. $1-2x+4x^2 - \dfrac{6x^3-4x^4}{1+x-x^2} = \dfrac{1-x+x^2}{1+x-x^2}$.

Ex. 15. $$\frac{a+b}{a-b} - \frac{a^2-b^2}{a^2+b^2} + \frac{a^2+b^2}{a^2-b^2}$$
$$= \frac{a^4+2a^3b+6a^2b^2+2ab^3+b^4}{a^4-b^4}.$$

Ex. 16. $$\frac{1}{x-1} - \frac{1}{2x+2} - \frac{x+3}{2x^2+2} = \frac{x+3}{x^4-1}.$$

Ex. 17. $$\frac{3h}{(h-2x)^2} + \frac{2h+x}{(h+x)(h-2x)} - 5\frac{1}{h+x}$$
$$= \frac{20hx-22x^2}{(h+x)(h-2x)^2}.$$

Ex. 18. $$\frac{a^n}{2na^n-2nx} + \frac{b^n}{2nb^n-2nx} = \frac{1}{n}, \text{ if } x = \frac{1}{2}(a^n+b^n).$$

Ex. 19. $$\frac{1}{n-1-(n-1)x} - \frac{1}{n+1+(n+1)x}$$
$$= \frac{2(nx+1)}{(n^2-1)(1-x^2)}.$$

Ex. 20. $$\frac{1}{4a^2(x-a)} - \frac{1}{4a^2(x+a)} - \frac{x}{2a^2(x^2+a^2)} = \frac{x}{x^4-a^4}.$$

Ex. 21. $$\frac{x}{x-3} - \frac{x-3}{x} + \frac{x}{x+3} - \frac{x+3}{x} = \frac{18}{x^2-9}.$$

Ex. 22. $$\frac{1}{8(x-1)} - \frac{1}{4(x-3)} + \frac{1}{8(x-5)}$$
$$= \frac{1}{(x-1)(x-3)(x-5)}.$$

Ex. 23. $$\frac{1}{(x+1)(x+2)} - \frac{1}{(x+1)(x+2)(x+3)}$$
$$= \frac{1}{(x+1)(x+3)}.$$

Ex. 24. $$\frac{1}{x^3} + \frac{1}{x^2} - \frac{1}{x} - \frac{1}{(x^2+1)^2} + \frac{x-1}{x^2+1} = \frac{x^2+x+1}{x^3(x^2+1)^2}.$$

Ex. 25. $$1-x+x^2-x^3+\frac{x^4}{1+x} = \frac{1}{1+x}.$$

MULTIPLICATION.

ART. 35. To find the product of two fractions.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any 2 fractions, and let $\frac{a}{b} = x, \frac{c}{d} = y$;

$$\therefore\ a = bx,\ c = dy;$$

$$\therefore\ ac = bdxy;$$

$$\therefore\ xy, \text{ or } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Hence the following Rule. Multiply the numerators together for a new numerator, and the denominators for a new denominator.

Ex. 1. Multiply $\frac{a+b}{c}$ by $\frac{a-b}{d}$.

$$\frac{a+b}{c} \times \frac{a-b}{d} = \frac{(a+b)(a-b)}{cd} = \frac{a^2-b^2}{cd}.$$

Ex. 2. Multiply $a+x$ by $\frac{3d}{c}$.

$$(a+x) \times \frac{3d}{c} = \frac{a+x}{1} \times \frac{3d}{c} = \frac{3ad+3dx}{c}.$$

Ex. 3. Multiply $b + \frac{bx}{a}$ by $\frac{a}{x}$.

$$\left(b + \frac{bx}{a}\right) \times \frac{a}{x} = \frac{ab+bx}{a} \times \frac{a}{x} = \frac{ab+bx}{x}.$$

In this Example, the a is cancelled in the numerator and denominator.

Ex. 4. Multiply $\frac{a^2}{x^2} - \frac{ab}{2xy} + \frac{b^2}{y^2}$ by $\frac{3a^2}{x^2} - \frac{2ab}{5xy} + \frac{b^2}{y^2}$.

$$\begin{array}{rrrrr}
\dfrac{a^2}{x^2} - & \dfrac{ab}{2xy} + & \dfrac{b^2}{y^2} & & \\
\dfrac{3a^2}{x^2} - & \dfrac{2ab}{5xy} + & \dfrac{b^2}{y^2} & & \\
\hline
\dfrac{3a^4}{x^4} - & \dfrac{3a^3b}{2x^3y} + & \dfrac{3a^2b^2}{x^2y^2} & & \\
 - & \dfrac{2a^3b}{5x^3y} + & \dfrac{a^2b^2}{5x^2y^2} - & \dfrac{2ab^3}{5xy^3} & \\
 & & \dfrac{a^2b^2}{x^2y^2} - & \dfrac{ab^3}{2xy^3} + & \dfrac{b^4}{y^4} \\
\hline
\dfrac{3a^4}{x^4} - & \dfrac{19a^3b}{10x^3y} + & \dfrac{21a^2b^2}{5x^2y^2} - & \dfrac{9ab^3}{10xy^3} + & \dfrac{b^4}{y^4}
\end{array}$$

This same result would have been obtained, if we had first reduced the fractions in the multiplier and multiplicand to a common denominator, and then multiplied them together, and rejected common factors from the numerator and denominator of the product.

Ex. 5. $\dfrac{a^2 - b^2}{5b} \times \dfrac{15a^2}{a+b} = \dfrac{3a^2(a-b)}{b}$.

Ex. 6. $\dfrac{x+1}{x-1} \times \dfrac{x^2+x-2}{x^2-x} = \dfrac{x^2+3x+2}{x^2-x}$.

Ex. 7. $\dfrac{a}{b} - \dfrac{c}{b} \times \dfrac{ae-bd}{ce-bf} = \dfrac{cd-af}{ce-bf}$.

Ex. 8. $\left(a - \dfrac{x^2}{a}\right) \times \left(\dfrac{a}{x} + \dfrac{x}{a}\right) = \dfrac{a^4-x^4}{a^2x}$.

Ex. 9. $\dfrac{ax}{(a-x)^2} \times \dfrac{a^2-x^2}{ab} = \dfrac{ax+x^2}{ab-bx}$.

Ex. 10. $\dfrac{5b}{a^2-b^2} \times \dfrac{a+b}{15a^2} = \dfrac{b}{3a^2(a-b)}$.

Ex. 11. $\dfrac{a^3-b^3}{a^3+b^3} \times \dfrac{(a+b)^2}{(a-b)^2} = \dfrac{a^3+2a^2b+2ab^2+b^3}{a^3-2a^2b+2ab^2-b^3}$.

Ex. 12. $\left(\dfrac{a}{x} + \dfrac{b}{y}\right) \times \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) = \dfrac{x}{a} + \dfrac{bx^2}{a^2y} + \dfrac{ay^2}{b^2x} + \dfrac{y}{b}$.

Ex. 13. $\dfrac{2a-x}{(c-x)^2} \times \dfrac{cx-x^2}{c^2+cx+x^2} = \dfrac{2ax-x^2}{c^3-x^3}$.

Ex. 14. $\left(\dfrac{4a}{3x} + \dfrac{3x}{2b}\right) \times \left(\dfrac{2b}{3x} + \dfrac{3x}{4a}\right) = \dfrac{8ab}{9x^2} + 2 + \dfrac{9x^2}{8ab}$.

Ex. 15. $\left\{\dfrac{a}{a-b} + \dfrac{b}{a+b}\right\} \times \left\{\dfrac{a}{a-b} - \dfrac{b}{a+b}\right\}$

$$= \frac{a^3(a+2b)-b^3(b-2a)}{(a^2-b^2)^2}.$$

Ex. 16. $\dfrac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3} \times \dfrac{a^2-ax+x^2}{a+x} = \dfrac{a^4+a^2x^2+x^4}{a^4-x^4}$.

Ex. 17. $\dfrac{x}{x+a} \times \dfrac{2x}{a^2-ax+x^2} = \dfrac{2x^2}{a^3+x^3}$.

Ex. 18. $\dfrac{x^2-9x+20}{x^2-6x} \times \dfrac{x^2-13x+42}{x^2-5x} = \dfrac{x^2-11x+28}{x^2}$.

Ex. 19. $\dfrac{a^2-b^2}{x+y} \times \dfrac{x^2-y^2}{a-b} \times \dfrac{a^2}{(x-y)^2} = \dfrac{a^2(a+b)}{x-y}$.

Ex. 20. Multiply $15a^{-6}b^2 - 7a^{-5}b^4 + 6a^{-4}b^6$ by $8a^{-2}b^2 - 3a^{-1}b^4$.
Ans. $120a^{-8}b^4 - 101a^{-7}b^6 + 69a^{-6}b^8 - 18a^{-5}b^{10}$.

Ex. 21. $\left(\dfrac{x}{a} + \dfrac{a}{x}\right)\left(\dfrac{y}{b} + \dfrac{b}{y}\right) + \left(\dfrac{x}{a} - \dfrac{a}{x}\right)\left(\dfrac{y}{b} - \dfrac{b}{y}\right)$

$$= \frac{2xy}{ab} + \frac{2ab}{xy}.$$

Ex. 22. $\dfrac{a-x}{a+x} - \dfrac{ax}{a^2-x^2} \times \dfrac{(a-x)(a+3x)}{a-3x}$

$$= \frac{a^3-8ax^2+3x^3}{a^2-2ax-3x^2}.$$

DIVISION.

Art. 36. To find the quotient of two fractions.

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be any 2 fractions, and let $\dfrac{a}{b} = x$, $\dfrac{c}{d} = y$;

$$\therefore\ a = bx,\ c = dy;$$

$$\therefore\ ad = bdx,\ bc = bdy;$$

$$\therefore\ \frac{bdx}{bdy} \text{ or } \frac{x}{y} = \frac{ad}{bc};$$

$$\therefore\ \frac{x}{y} \text{ or } \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}.$$

Hence the following Rule. Invert the divisor and proceed as in multiplication.

Ex. 1. Divide $\frac{a^2-b^2}{cd}$ by $\frac{a+b}{c}$.

$$\frac{a^2-b^2}{cd} \div \frac{a+b}{c} = \frac{a^2-b^2}{cd} \times \frac{c}{a+b} = \frac{(a-b)(a+b)c}{cd(a+b)}$$

$$= \frac{a-b}{d}, \text{ cancelling common factors.}$$

Ex. 2. Divide $\frac{a^2+ab}{2x}$ by $\frac{3a^2}{a-b}$.

$$\frac{a^2+ab}{2x} \div \frac{3a^2}{a-b} = \frac{a(a+b)}{2x} \times \frac{a-b}{3a^2}$$

$= \frac{a^2-b^2}{6ax}$, by cancelling a in the numerator and denominator.

Ex. 3. Shew that $a \div \frac{b}{x} = \frac{ax}{b}$.

$$\text{Let } \frac{b}{x} = y;$$

$$\therefore\ b = xy, \text{ and } ax = ax;$$

$$\therefore\ \frac{ax}{xy} = \frac{ax}{b};$$

$$\therefore\ \frac{a}{y} = \frac{ax}{b};$$

$$\text{or } a \div \frac{b}{x} = \frac{ax}{b};$$

$$\text{Similarly, } \frac{b}{x} \div a = \frac{b}{ax};$$

The same also appears from considering a as a fraction $\frac{a}{1}$.

Ex. 4. Divide $\frac{3a^4}{x^4} - \frac{19a^3b}{10x^3y} + \frac{21a^2b^2}{5x^2y^2} - \frac{9ab^3}{10xy^3} + \frac{b^4}{y^4}$ by $\frac{3a^2}{x^2} - \frac{2ab}{5xy} + \frac{b^2}{y^2}$.

Arranging them as in long division, we have

$$\begin{array}{l}
\frac{3a^2}{x^2} - \frac{2ab}{5xy} + \frac{b^2}{y^2}\Big) \\
\qquad \frac{3a^4}{x^4} - \frac{19a^3b}{10x^3y} + \frac{21a^2b^2}{5x^2y^2} - \frac{9a^3b}{10xy^3} + \frac{b^4}{y^4}\Big(\frac{a^2}{x^2} - \frac{ab}{2xy} + \frac{b^2}{y^4} \\
\qquad \frac{3a^4}{x^4} - \frac{2a^3b}{5x^3y} + \frac{a^2b^2}{x^2y^2} \\
\hline
\qquad -\frac{3a^3b}{2x^3y} + \frac{16a^2b^2}{5x^2y^2} - \frac{9ab^3}{10xy^3} \\
\qquad -\frac{3a^3b}{2x^3y} + \frac{a^2b^2}{5x^2y^2} - \frac{ab^3}{2xy^3} \\
\hline
\qquad \frac{3a^2b^2}{x^2y^2} - \frac{2ab^3}{5xy^3} + \frac{b^4}{y^4} \\
\qquad \frac{3a^2b^2}{x^2y^2} - \frac{2ab^3}{5xy^3} + \frac{b^4}{y^4} \\
\hline
\end{array}$$

The same result would have been obtained, if we had first reduced the fractions in the divisor and dividend to a common denominator, and after division, rejected common factors from the numerator and denominator of the quotient.

Ex. 5. Divide $-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8$ by $2a^{-3}x^3 - 3ax^4$.

$$\begin{array}{l}
2a^{-3}x^3 - 3ax^4) - 2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8(-a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4 \\
\qquad\qquad\qquad - 2a^{-8}x^5 + 3a^{-4}x^6 \\
\hline
\qquad\qquad\qquad 14a^{-4}x^6 - 5x^7 \\
\qquad\qquad\qquad 14a^{-4}x^6 - 21x^7 \\
\hline
\qquad\qquad\qquad\quad 16x^7 - 24a^4x^8 \\
\qquad\qquad\qquad\quad 16x^7 - 24a^4x^8 \\
\hline
\end{array}$$

Or we may write the divisor and dividend in a fractional form previous to division. Thus the divisor is $\frac{2x^3}{a^3} - 3ax^4$, and the dividend is $-\frac{2x^5}{a^8} + \frac{17x^6}{x^4} - 5x^7 - 24a^4x^8$.

Ex. 6. $\frac{3a^2x}{2b^3} \div \frac{ab}{4x} = \frac{6ax^2}{b^4}$ and $\frac{a^2-b^2}{a^2+2b} \div \frac{a-b}{3a+6b} = 3(a+b)$.

Ex. 7. $\frac{2x^2}{a^3+x^3} \div \frac{x}{x+a} = \frac{2x}{a^2-ax+x^2}$.

Ex. 8. $\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+bx}{x-b} = \frac{x^2+b^2}{x}$.

Ex. 9. $\frac{x^2+5x+4}{x^2+7x+12} \div \frac{x^2+2x+1}{x^2+3x+2} = \frac{x+2}{x+3}$.

Ex. 10. $\frac{a+b}{a^2+2b^2} \div \frac{a^2-2b^2}{a-b} = \frac{a^2-b^2}{a^4-4b^4}$.

Ex. 11. $\frac{x^2+xy}{x-y} \div \frac{x^4-y^4}{(x-y)^2} = \frac{x}{x^2+y^2}$.

Ex. 12. $\frac{2a(1-x^2)^2}{cy} \div \frac{(1-x)(1+x)^2}{y^3} = \frac{2ay^2(1-x)}{c}$.

Ex. 13. $\left\{x^4 - \frac{a^2-x^2}{4} + a^3x - a^4\right\} \div \left\{x^2 - \frac{ax}{2} + a^2\right\} = x^2 + \frac{ax}{2} - a^2$.

Ex. 14. $\frac{2ax-x^2}{b^3-x^3} \div \frac{2a-x}{(b-x)^2} = \frac{bx-x^2}{b^2+bx+x^2}$.

Ex. 15. $\left\{\frac{8ab}{9x^2} + 2 + \frac{9x^2}{8ab}\right\} \div \left\{\frac{4a}{3x} + \frac{3x}{2b}\right\} = \frac{2b}{3x} + \frac{3x}{4a}$.

Ex. 16. $\left\{y^3 - \frac{1}{y^3} - 3\left(y - \frac{1}{y}\right)\right\} \div \left(y - \frac{1}{y}\right) = \left(y - \frac{1}{y}\right)^2$.

Ex. 17. $(a+x)^2(a-y)^{-3} \div (a+x)^{-4}(a-y)^{-7} = (a+x)^6(a-y)^4$.

Ex. 18. $\frac{x^3+2x^2-x-2}{x^3-2x^2+x} \div \frac{x+1}{x-1} = \frac{x^2+x-1}{x^2-x}$.

Ex. 19.

$$\left\{\frac{a}{ab} + \frac{b}{a+b}\right\} \div \left\{\frac{a}{a-b} - \frac{b}{a+b}\right\} = \frac{(a+b+b^2)(a-b)}{b(a^2+b^2)}.$$

Ex. 20. $(a^4b^{-6} - 4c^6d^8b^{-10} + 14c^5d^4a^{-4}b^{-8} - \frac{49}{4}c^4a^{-8}b^{-6}) \div (a^2b^{-3} - 2c^3d^4b^{-5} + \frac{7}{2}c^2a^{-4}b^{-3}) = a^2b^{-3} + 2c^3d^4b^{-5} - \frac{7}{2}c^2a^{-4}b^{-3}$.

Ex. 21.

$$\left\{x^4 - \frac{x}{2} + \frac{3x^2}{2} + \frac{1}{16} - 2x^3\right\} \div \left(x^2 - x + \frac{1}{4}\right) = x^2 - x + \frac{1}{4}.$$

Ex. 22.

$$\left\{x^2 + \frac{1}{x^2} + 2\left(x - \frac{1}{x}\right) - 1\right\} \div \left\{x - \frac{1}{x} + 1\right\} = x - \frac{1}{x} + 1.$$

Ex. 23.

$$\left\{\frac{a}{b} + \frac{c}{d} + \frac{e}{f}\right\} \div \left\{\frac{l}{m} + \frac{n}{r} + \frac{s}{t}\right\} = \frac{(adf + bcf + bde)\,mrt}{(lrt + mnt + mrs)\,bdf}.$$

Ex. 24. $$\frac{1}{\dfrac{1}{x-a} + \dfrac{1}{x-b}} = \frac{x^2 - (a+b)\,x + ab}{2x - a - b}.$$

Ex. 25. $$\frac{1}{\dfrac{1}{x+2} + \dfrac{1}{x+6} + \dfrac{1}{x+8}} = \frac{x^3 - 52x - 96}{3x^2 - 52}.$$

Ex. 26. $$\frac{\dfrac{1}{1-x}}{\dfrac{1}{1-x} - 1} = \frac{1}{x}.$$

Ex. 27. $$\left\{\frac{a+x}{a-x} + \frac{a-x}{a+x}\right\} \div \left\{\frac{a+x}{a-x} - \frac{a-x}{a+x}\right\} = \frac{a^2 + x^2}{2ax}.$$

Ex. 28. $$\frac{1}{1 + \dfrac{1}{x + \dfrac{1}{x}}} = \frac{1 + x^2}{1 + x + x^2}.$$

Ex. 29. Divide $x^3 - 12x^2 + 54x - 112 + \dfrac{108}{x} - \dfrac{48}{x^2} + \dfrac{8}{x^3}$, by $\left(x + \dfrac{2}{x} - 4\right)^2$. Ans. $x + \dfrac{2}{x} - 4$.

Ex. 30. Find the value of $\dfrac{x-1}{x+1+\dfrac{x-2}{x+2+\dfrac{x-3}{x+3}}}$.

Ans. $$\frac{x^3 + 5x^2 - 3x - 3}{x^3 + 8x^2 + 10x - 3}.$$

Ex. 31. Shew that $\dfrac{x+3}{2} - \dfrac{11-x}{5} = \dfrac{3x-1}{20} + 3\tfrac{1}{5}$ when $x = 7$,

and $\dfrac{4x+3}{9} + \dfrac{7x-29}{5x-12} = \dfrac{8x+19}{18}$ when $x = 6$.

Ex. 32. Shew that $(a+x)(b+x) - a(b+c) = \dfrac{a^2c}{b} + x^2$, if $x = \dfrac{ac}{b}$,

$$\text{and } \frac{5x-9}{3+\sqrt{5x}} - 1 = \frac{1}{2}\left\{\sqrt{5x} - 3\right\}, \text{ if } x = 5.$$

Ex. 33. Shew that $2\sqrt{x} - \sqrt{4x = \sqrt{7x+2}} = 1$, if $x = 1$.

$$\text{and } \frac{5(3x-1)}{1+5\sqrt{x}} + \frac{2}{\sqrt{x}} = 6 \text{ if } x = 4.$$

Ex. 34. Shew that

$$\frac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}} = 4\tfrac{1}{4} - \frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}} \text{ if } x = 5,\ y = 4,$$

$$\text{and } \frac{2x+y^{\frac{1}{2}}}{2x-y^{\frac{1}{2}}} = \frac{16}{15} + \frac{2x-y^{\frac{1}{2}}}{2x+y^{\frac{1}{2}}}, \text{ if } x = 2,\ y = 1.$$

CHAPTER IV.

RATIO, PROPORTION, AND VARIATION.

ART. 37. *Ratio* is the relation which exists between two quantities of the same kind with respect to magnitude, the comparison being made by considering what multiple, part or parts, one of them is of the other.

Thus, if a and b be any two quantities whatever of the same kind, their *ratio* will be represented by the fraction $\frac{a}{b}$, which indicates what multiple, part or parts, a is of b.

This ratio of a to b, or the fraction $\frac{a}{b}$, is frequently written thus, $a : b$, and the former term a is called the *antecedent* of the ratio, and the latter b the *consequent*.

Ratios are compared by comparing the vulgar fractions which express their values. Thus the ratio of a to b is lesser than, or equal to, or greater than, the ratio of c to d, according as $\frac{a}{b}$ is $<$ = or $>$ $\frac{c}{d}$.

ART. 38. *Proportion* is the relation of equality subsisting between two ratios. Thus, if a, b, c, d, be four quantities, since the ratio of the first two is $\frac{a}{b}$, and of the latter $\frac{c}{d}$, the equivalence of these ratios, or the proportionality of a, b, c, d, is expressed by the equation, $\frac{a}{b} = \frac{c}{d}$. This proportion is also written thus, $a : b = c : d$, or $a : b :: c : d$, and read thus, as a is to b so is c to d. The terms a and d are called the *extremes*, and b and c the *means*.

Since $\frac{12}{4} = 3$ and $\frac{15}{5} = 3$; $\therefore \frac{12}{4} = \frac{15}{5}$ or 12, 4, 15, and 5 are in proportion. Similarly, na, a, nb, b, are proportionals.

ART. 39. If $a : b :: c : d$, then $ad = bc$, and the converse.

$$\text{Since } a : b :: c : d;$$

$$\therefore \frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a}{b} \times bd = \frac{c}{d} \times bd;$$

$$\therefore ad = bc,$$

or the product of the *extremes* equals the product of the *means*.

Again, let $ad = bc$.

$$\therefore \frac{ad}{bd} = \frac{bc}{bd};$$

$$\therefore \frac{a}{b} = \frac{c}{d};$$

$$\text{and therefore } a : b :: c : d.$$

ART. 40. If a, b, c, be in *continued* proportion, so that

$$a : b :: b : c,$$

$$\text{then (Art. 39.) } ac = b^2,$$

$$\text{and similarly, if } ac = b^2,$$

$$a : b :: b : c.$$

ART. 41. By means of Art. 39, we are enabled when any three terms of a proportion are given, to find the fourth.

Thus, let a, b, c, be the three first terms, and let it be required to find x the fourth term. Then

$$a : b :: c : x;$$

$$\therefore ax = bc;$$

$$\therefore x = \frac{bc}{a},$$

which demonstrates the Rule of Three in Arithmetic. And similarly, if any other three terms of the proportion be given, the remaining one may be found.

ART. 42. If $a : b :: c : d$,

then $b : a :: d : c$; $a : c :: b : d$; and $a + b : b :: c + d : d$.

$$\text{Since } a : b :: c : d,$$

$$\therefore \frac{a}{b} = \frac{c}{d};$$

$$\therefore\ 1 \div \frac{a}{b} = 1 \div \frac{c}{d},$$

$$\text{or } \frac{b}{a} = \frac{d}{c};$$

that is, $b : a :: d : c$, (Invertendo.)

$$\text{Also, since } \frac{a}{b} = \frac{c}{d};$$

$$\therefore\ \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c},$$

$$\text{or } \frac{a}{c} = \frac{b}{d};$$

that is, $a : c :: b : d$. (Alternando.)

$$\text{Again, since } \frac{a}{b} = \frac{c}{d},$$

$$\therefore\ \frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\text{or } \frac{a+b}{b} = \frac{c+d}{d},$$

that is, $a+b : b :: c+d : d$. (Componendo.)

ART. 43. If $a : b :: c : d$, and $c : d :: e : f$,

then $a : b :: e : f$.

Since $a : b :: c : d$,

$$\therefore\ \frac{a}{b} = \frac{c}{d}.$$

Also, since $c : d :: e : f$;

$$\therefore\ \frac{c}{d} = \frac{e}{f}.$$

$$\text{But } \frac{a}{b} = \frac{c}{d};$$

$$\therefore\ \frac{a}{b} = \frac{e}{f},$$

or $a : b :: e : f$.

ART. 44. If $a : b :: c : d$, and $b : e :: d : f$,

then $a : e :: c : f$.

Since $a : b :: c : d$;

$$\therefore\ \frac{a}{b} = \frac{c}{d}.$$

Also since $b : e :: d : f$;

$$\therefore \frac{b}{e} = \frac{d}{f};$$

$$\therefore \frac{a}{b} \times \frac{b}{e} = \frac{c}{d} \times \frac{d}{f};$$

$$\text{or } \frac{a}{e} = \frac{c}{f};$$

that is, $a : e :: c : f$.

ART. 45. The following is the *Geometrical* definition of proportion.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equi-multiples whatsoever of the first and third being taken, and any equi-multiples whatsoever of the second and fourth; if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth: or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth: or, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth. (Euc. Book 5. Def. 5.)

ART. 46. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

Let a, b, c, d, be proportional according to the *algebraical* definition; that is, let

$$a : b :: c : d,$$

$$\text{or } \frac{a}{b} = \frac{c}{d};$$

$\therefore$ multiplying each side by $\frac{m}{n}$, we have

$$\frac{ma}{nb} = \frac{mc}{nd}.$$

Hence, if $ma >=$ or $< nb$, mc will be $>=$ or $< nd$.

And because there are four magnitudes a, b, c, d, and of a and c, the first and third, any equi-multiples whatever have

been taken, *viz.* ma and mc, and of b and d, the second and fourth, any equi-multiples whatever have been taken, *viz.* nb and nd, and because as ma is $>=$ or $< nb$, so is $mc >=$ or $< nd$, therefore a, b, c, d are proportional according to the *geometrical* definition.

ART. 47. A quantity is said to *vary* as another, when it is so dependent upon it, that every change which the latter undergoes, produces a corresponding and proportional change in the magnitude of the former.

A quantity is said to vary *directly* as another when it increases or decreases according as the other increases or decreases.

A quantity is said to vary *inversely* as another when it increases or decreases according as the other decreases or increases.

A quantity is said to vary as two others *jointly*, when it varies *directly* as their product.

If A varies *directly* as B, it is written $A \propto B$, or $A = pB$; if *inversely* as B, it is written $A \propto \frac{1}{B}$, or $A = \frac{p}{B}$; if *jointly*, as B and C, it is written $A \propto BC$, or $A = pBC$, p being a factor which undergoes no change, however the magnitudes A, B, C, may vary.

Similarly, if A varies as B *directly*, and C *inversely*, it is written $A \propto \frac{B}{C}$, or $A = p\,\frac{B}{C}$.

ART. 48. Ex. 1. It is required to find x, when 3, x and 1083 form a continued proportion.

By the question (Vid. Art. 40.)

$$3 : x :: x : 1083;$$

$$\therefore\ 3 \times 1083, \text{ or } 3249 = x^2;$$

$$\therefore\ x = \sqrt{3249} = 57.$$

Ex. 2. If $a : b :: c : d$, then $a - b : b :: c - d : d$.

$$\frac{a}{b} = \frac{c}{d},$$

$$\therefore\ \frac{a}{b} - 1 = \frac{c}{d} - 1;$$

$$\therefore \frac{a-b}{b} = \frac{c-d}{d};$$

$$\therefore a-b : b :: c-d : d. \text{ (Dividendo.)}$$

Ex. 3. If $a : b :: c : d$, then $a - b : a :: c - d : c$. (Convertendo.)

Ex. 4. If $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

Ex. 5. If $a : b :: c : d$, then $ma : mb :: nc : nd$, and $a^n : b^n :: c^n : d^n$.

Ex. 6. If $a : b :: c : d :: e : f$, then $\frac{a}{m} : b :: \frac{c}{m} : d$, and $a : b :: a + c + e : b + d + f$.

$$\text{For } \frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a}{mb} = \frac{c}{md};$$

$$\therefore \frac{\frac{a}{m}}{b} = \frac{\frac{c}{m}}{d};$$

$$\therefore \frac{a}{m} : b :: \frac{c}{m} : d.$$

$$\text{Again } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{a}{b} = \frac{e}{f};$$

$$\therefore ad = bc \text{ and } af = be;$$

$$\therefore ab + ad + af = ab + bc + be;$$

$$\text{or, } a(b + d + f) = b(a + c + e);$$

$$\therefore \frac{a}{b} = \frac{a+c+e}{b+d+f},$$

$$\text{or, } a : b :: a + c + e : b + d + f.$$

Ex. 7. If 4, x and 36 form a continued proportion, and 2, y and 98 form a continued proportion, find the values of x and y. Ans. $x = 12$, $y = 14$.

Ex. 8. If $y = mx$, where m is constant, then y varies as x.

$$y = mx.$$

Now if m and x be both variable, the value of y will alter by

giving different values to m or x. But in this case m is constant, and therefore the value of y can only alter by giving different values to x, or

$$y \propto x.$$

Obs. This will immediately appear if we put any numerical value, as 5 for m, in which case $y = 5x$, or $y \propto x$.

Ex. 9. If $A \propto B$, and $B \propto C$, then $A \propto C$.

Since $A \propto B$, let $A = pB$;

.... $B \propto C$, let $B = qC$;

$\therefore\ A = pB = pqC$, or $A \propto C$.

Ex. 10. If $A \propto \frac{1}{B}$ and $B \propto C$, then $A \propto \frac{1}{C}$.

Ex. 11. If $A \propto \frac{1}{B}$ and $B \propto \frac{1}{C}$, then $A \propto C$.

$$\text{Let } A = \frac{p}{B}, \text{ and } B = \frac{q}{C};$$

$$\therefore\ A = \frac{p}{B} = p \div \frac{q}{C} = \frac{p}{q}C,$$

or $A \propto C$.

Ex. 12. If $x - a \propto y - b$, and when $x = m$, $y = n$; find the constant which connects $x - a$ with $y - b$.

Since $x - a \propto y - b$,

let $x - a = p(y - b)$, p being the constant to be determined.

Now when $x = m$, $y = n$;

$$\therefore\ m - a = p(n - b);$$

$$\therefore\ p = \frac{m-a}{n-b}.$$

Ex. 13. If $y^2 \propto a^2 - x^2$ and when $x = \sqrt{a^2 - b^2}$, $y = \frac{b^2}{a}$ find the equation between x and y. Ans. $y^2 = \frac{b^2}{a^2}(a^2 - x^2.)$

Ex. 14. If $y = p + q$, where $p \propto x$, and $q \propto \frac{1}{x}$; also when $x = 1$, $y = 6$, and when $x = 2$, $y = 5$, then $y = \frac{4}{3}x + \frac{14}{3}x$.

$$\text{Let } p = cx,\ q = \frac{d}{x};$$

$$\therefore\ y = cx + \frac{d}{x}.$$

But when $x = 1$, $y = 6$;

$$\therefore\ 6 = c + d \ldots\ldots\ldots\ldots \quad (1)$$

and when $x = 2$, $y = 5$;

$$\therefore\ 5 = 2c + \frac{d}{2} \ldots\ldots\ldots\ldots \quad (2)$$

Now from equation (1) $12 = 2c + 2d$, multiplying by 2;

.... (2) $5 = 2c + \dfrac{d}{2}$;

$$\therefore \text{ by subtraction } 7 = \frac{3d}{2}, \text{ or } d = \frac{14}{3};$$

$$\text{and from (1) } c = 6 - d = 6 - \frac{14}{3} = \frac{18}{3} - \frac{14}{3} = \frac{4}{3};$$

$$\therefore\ y = \frac{4}{3}x + \frac{14}{3x}.$$

Ex. 15. Find a fourth proportional to $\frac{2}{7}$, $\frac{3}{4}$, and $\frac{5}{6}$.

Let x be the proportional, then

$$\tfrac{2}{7} : \tfrac{3}{4} :: \tfrac{5}{6} : x;$$

$$\therefore\ \frac{2x}{7} = \frac{15}{24};$$

$$\therefore\ x = \frac{7}{2} \times \frac{15}{24} = 2\tfrac{3}{16}.$$

Ex. 16. Find a fourth proportional to ·0004, 1·4, ·02.

Ans. 70.

MISCELLANEOUS EXAMPLES.

ART. 49. Ex. 1. Multiply $2a^2-3ab+4b^2$ by $3a^2+2ab-b^2$, and divide a^3-b^3 by a^2+ab+b^2.

Ex. 2. Reduce to their simplest forms $\frac{a^2b}{c} \div \frac{abc}{e^2}$ and $\frac{a}{x}+\frac{2a+3x}{a+x}-\frac{3x^2}{a^2-x^2}$.

Ex. 3. A straight line is 3 ft. 4 ins. long; divide it into 2 parts, one of which shall be to the other as 5 : 7.

Ex. 4. Divide $6x^4-96$ by $3x-6$, and multiply x^2-xy+y^2 by $x+y$.

Ex. 5. Find the difference between $x+\frac{x-a}{x(x+a)}$ and $\frac{x+a}{x(x-a)}$.

Ex. 6. Divide

$$\frac{2x^2}{a^3+x^3} \text{ by } \frac{x}{a+x} \text{ and } \frac{x^4-b^4}{x^2-2bx+b^2} \text{ by } \frac{x^2+bx}{x-b}.$$

Ex. 7. If $a:b::c:d$ then cannot $a+x:b+x::c+x:d+x$.

Ex. 8. Shew that 20 is a fourth proportional to ·0014, 1·4 and ·02.

Ex. 9. Shew that ·051 is a mean proportional between ·017 and ·153.

Ex 10. If $a:b::c:d::e:f$, then $a-e:b-f::c:d$.

Ex. 11. If $x \propto y^2z$, and 1, 2, 3, be contemporary values of x, y, and z, then $x=\frac{y^2z}{12}$.

Ex. 12. If $ab \propto a^2+b^2$, and when $a=3$, $b=4$, shew that $ab=\frac{12}{25}(a^2+b^2)$.

Ex. 13. Shew that no power, or root of a real fraction can be a whole number.

Ex. 14. If $a+b \propto a-b$, then $a^2+b^2 \propto ab$.

Ex. 15. Prove the rule for Subtraction in Algebra, and simplify the following quantities,

$$(a+p)\,x^2-(px-b)\,x,\ \ a-\{a-(b+x)\}+(b-\overline{x-2b}).$$

Ex. 16. Simplify the following expressions:

$$(1)\quad \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}\text{ when } x=\frac{2ab}{b^2+1}.$$

$$(2)\quad \frac{8}{5(x-2)}-\frac{13}{80(x+3)}-\frac{5}{4(x-1)^2}-\frac{23}{16(x-1)}.$$

Ex. 17. Shew that $\dfrac{10x+17}{18}-\dfrac{12x+2}{11x-8}=\dfrac{5x-4}{9}$ when $x=4$.

Ex. 18. Prove fully that $(a-b)(c-d)=ac-bc-ad+bd$, and find the continued product of $(1-ax)(1-bx)(1-cx)$.

Ex. 19. Multiply $\dfrac{2x^2}{b}-\dfrac{3a^2}{b}$ by $\dfrac{2x^2}{c}+\dfrac{3a^2}{c}$; and find the least common multiple of $3a$, $7a^2$, $11ab$, and $21a^3$.

Ex. 20. Simplify the following expressions:

$$(1)\quad 3a-\{b+(2a-\overline{b-x})\}\qquad (2)\quad 2\,.\,\frac{x^2-\frac{1}{4}}{2x+1}+\tfrac{1}{2}.$$

$$(3)\quad \frac{\dfrac{1}{a}+\dfrac{1}{ab^3}}{b-1+\dfrac{1}{b}}.$$

Ex. 21. Divide $a'+x$ by $a+x$ as far as 4 terms, and restore the result with its remainder to the form $\dfrac{a'+x}{a+x}$.

Ex. 22. Divide $12a^3x-4axb-9a^4b^3+18a^2b^4-5b^5$ by $3a^2-2b$, and subtract $\dfrac{1-x^2}{1-x^2+x^3}$ from $\dfrac{1+x^2}{1+x^2+x^3}$.

APPENDIX.

Art. 50. To test the accuracy of the multiplication of numbers by *casting out the nines.*

Lemma. Any number divided by 9 will leave the same remainder as the sum of its digits divided by 9.

Let N and N' be two numbers containing p and p' nines respectively, with the remainders r, r', so that

$$N = 9p + r, \quad N' = 9p' + r';$$

$$\begin{aligned}\therefore\ NN' &= (9p + r)(9p' + r') \\ &= 81pp' + 9p'r + 9pr' + rr' \\ &= 9(9pp' + p'r + pr') + rr'; \\ \therefore\ \frac{NN'}{9} &= 9pp' + p'r + pr' + \frac{rr'}{9};\end{aligned}$$

and therefore NN' when divided by 9 leaves the same remainder as rr' divided by 9; or (Vid. Lemma, or Arithmetic, Art. 13. (6).) the sum of the digits of the product when divided by 9, leaves the same remainder, as the sum of those of the product of the partial remainders leaves.

Ex. Let $N = 3467$, and $N' = 692$, then $r = 2$, $r' = 8$,

also $NN' = 2399164$, and $rr' = 16$;

and the remainders arising from the division of the sums of the digits in NN' and rr' by 9 are both 7.

Obs. The above method of proof lies under the following disadvantages:

(1) If an error of 9 or any of its multiples be committed, the results will nevertheless agree; and so the error remain undetected.

(2) When it appears by the disagreement of the results, that an error has been committed, the particular figure or figures

in which the error lies are not pointed out; and consequently it is not easily corrected.

ART. 51. To find the greatest common measure of two quantities.

Lemma I. If any quantity measure another, it will also measure any multiple of that quantity. Thus, if a measure b by the units in p, or if $b = ap$, then it will measure qb by the units in pq, since $qb = apq$.

Lemma II. If a quantity measure two others, it will measure their sum and difference. Thus, if a measure b and c by the units in p and q, so that $b = ap$, $c = aq$, then

$$b \pm c = ap \pm aq = a(p \pm q)$$

or a measures their sum and difference by the units in $p \pm q$.

Let a and b be the two quantities, a being greater than b, and let b be contained p times in a with a remainder c; let c be contained q times in b with a remainder d, and let d be contained r times in c with no remainder, the operation being as follows:

$$\begin{array}{l} b)\ a\ (p \\ \underline{pb} \\ \quad c)\ b\ (q \\ \quad \underline{qc} \\ \qquad d)\ c\ (r \\ \qquad \underline{rd} \\ \qquad\ \ o \end{array}$$

then shall d, the last divisor, be the greatest common measure of a and b.

First, d is a common measure of a and b. For by supposition it measures c, and $\therefore qc$ (Lem. I.) and $\therefore qc + d$ (Lem. II.) or b. Again $\because$ it measures b, $\therefore$ it measures pb, and $\therefore pb + c$ or a. Hence d is a common measure of a and b.

Secondly, it is the greatest common measure of a and b. For $\because d$ is a common measure of a and b, every measure of d will be a common measure of a and b, and $\therefore$ the greatest measure of d will be the greatest common measure of a and b. Now the greatest measure of d is itself, $\therefore$ the greatest common measure of a and b is d.

Hence the rule for finding the greatest common measure of two numbers.

ART. 52. To find the least common multiple of two quantities.

Let a and b be the two quantities, and d their greatest common measure, such that $a = pd$, $b = qd$. Now since p and q have no common measure except unity, their least common multiple will be their product pq. Hence the least common multiple of pd and qd, or a and b, is evidently pqd.

$$\text{But } pqd = \frac{pd \times qd}{d} = \frac{a \times b}{d},$$

$$\text{or least common multiple} = \frac{\text{the product}}{\text{G. C. M.}}.$$

Hence the rule for finding the least common multiple of two numbers.

ART. 53. To prove the rule for the multiplication of decimals.

Let $.P$ be the multiplier, and $\cdot Q$ the multiplicand, and let them contain p and q decimal places respectively. Then they may be represented in the forms of vulgar fractions by

$$\frac{P}{10^p} \text{ and } \frac{Q}{10^q},$$

$\therefore$ their product $= \dfrac{P}{10^p} \times \dfrac{Q}{10^q} = \dfrac{PQ}{10^{p+q}}$, or the product contains $p + q$ decimal places.

Hence, the multiplication of decimals is performed as in whole numbers, and the product contains as many decimal places as the multiplier and multiplicand together.

ART. 54. To prove the rule for the division of decimals.

By last Art. we shall have their quotient

$$= \frac{P}{10^p} \div \frac{Q}{10^q} = \frac{P}{10^p} \times \frac{10^q}{Q} = \frac{P}{Q} \times \frac{10^q}{10^p}.$$

(1) Let $p > q$, then the quotient is $\dfrac{P}{Q} \cdot \dfrac{1}{10^{p-q}}$, or, after the division is effected as in integers, the quotient will contain $p - q$ decimal places.

(2) Let $p = q$, then the quotient is $\dfrac{P}{Q}$, which is a whole number, if P be divisible by Q without a remainder.

(3) Let $p < q$, then the quotient is $\frac{P}{Q} \times 10^{q-p}$, or we must annex to the quotient $q-p$ cyphers, and the result will be a whole number.

ART. 55. To prove the rule for finding the value of a recurring decimal.

(1) Let it be a *pure* circulator as $\cdot PPP\ldots$ where P contains p digits.

$$\text{Let } x = \cdot PPP\ldots$$
$$\therefore\ 10^p . x = P.PP\ldots$$
$$\therefore \text{ by subtraction } (10^p - 1)\, x = P;$$
$$x = \frac{P}{10^p - 1}.$$

(2) Let it be a *mixed* circulator as $\cdot PQQ\ldots$ where P and Q contain p and q digits respectively.

$$\text{Let } x = \cdot PQQQ\ldots$$
$$\therefore\ 10^{p+q} x = PQ.QQ\ldots$$
$$\text{and } 10^p x = P.QQ\ldots$$
$$\therefore\ (10^{p+q} - 10^p)\, x = PQ - P;$$
$$\therefore\ x = \frac{PQ - P}{10^{p+q} - 10^p} = \frac{PQ - P}{10^p\,(10^q - 1)}.$$

And hence the rules given in Arithmetic.

ART. 56. No vulgar fraction in its lowest terms can be converted into a finite decimal, unless it be of the form $\frac{a}{2^p\, 5^q}$.

For the fraction being in its lowest terms, and the only factors of 10 and its powers being 2 and 5 and their powers, it is obvious, that if cyphers be affixed to the numerator, the result of the division by the denominator can never terminate, unless that denominator be composed of powers of one or both of the numbers 2 and 5, or unless the fraction be of the form $\frac{a}{2^p . 5^q}$.

The result will consist of p or q decimal places according as p is $>$ or $<$ q.

Obs. The *denary*, or common scale of notation, does not possess so many advantages as the *senary* or the scale which has 6 for its basis. For in the senary scale, the form of a fraction that is convertible into a finite decimal is $\frac{a}{2^p \,.\, 3^q}$, and it is evident that there are more multiples of 3 and its powers, than there are of 5 and its powers; in other words, more fractions could be expressed as terminating decimals in this scale, than in the common scale.

The reason of the adoption of the denary scale is probably to be found in the fact, that computation was first conducted by means of the ten fingers, and hence the term *digits* has been given to the figures commonly used.

Art. 57. Reason of the rule for the extraction of the square root in Arithmetic.

Since the square of $a + b$ is $a^2 + 2ab + b^2$; it is evident that the square root of $a^2 + 2ab + b^2$ is $a + b$.

Now in order to find how this root is obtained, we may observe, first, that the square root of the first term a^2 is a; and in addition to this there still remains $2ab + b^2$, from which we have to obtain b. Now the remainder $2ab + b^2 = (2a + b)\,b$, and therefore b will be obtained by dividing the first term of the remainder $2ab$ by $2a$, or *twice* the first term of the root; and then twice this first term together with the second, which completes the divisor, or $2a + b$, must be multiplied by the second term b, and after subtraction there will be no remainder. When the root consists of more than two terms, the process must be repeated.

Extract the square root of 4096.

$$
\begin{array}{rl}
 & 4096\ (\overset{a}{60} + \overset{b}{4} = 64 \\
 & \underline{3600} = a^2 \\
2 \times 60 + 4 = 124) & 496 = (2a + b)\,b \\
 & \underline{496}
\end{array}
$$

or omitting the cyphers, as in common practice,

$$\begin{array}{r|l} & \dot{4}0\dot{9}6\ (64 \\ & 36 \\ \hline 124) & 496 \\ & 496 \end{array}$$

ART. 58. Reason of the rule for the extraction of the cube root, in Arithmetic.

Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, it is evident that the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$.

Now in order to find how this root is obtained, we may observe first, that the cube root of the first term a^3 is a, and in addition to this, there still remains $3a^2b+3ab^2+b^3$, from which we have to obtain b. Now $3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)\,b$, and therefore b will be obtained by dividing the first term of the remainder $3a^2b$ by $3a^2$, or *thrice* the square of the first term of the root. The divisor $3a^2+3ab+b^2$ is then completed by adding to $3a^2$, *three* times the product of the two terms or $3ab$, and also the square of the last b^2. This divisor is then multiplied by b, and after subtraction there will be no remainder; and similarly if there be more than three terms in the root.

Extract the cube root of 2197.

$$\begin{array}{rr} & 2197\ (\overset{a}{10}+\overset{b}{3}=13 \\ a^3= & 1000 \\ \hline 3ab=\ 90 \qquad 3a^2=300) & 1197 \\ b^2=\ \ 9 \qquad\qquad\qquad & 1197 \\ \hline 3a^2+3ab+b^2=399 \quad \text{therefore } (3a^2+3ab+b^2)\,b=1197 & \end{array}$$

or as in common practice,

$$\left.\begin{array}{l} \text{first part} = 300 \\ \text{second part} = 90 \\ \text{third part} = 9 \end{array}\right\} \text{whole divisor} = 399) \begin{array}{l} \dot{2}19\dot{7}\ (13 \\ 1 \\ \hline 1197 \\ 1197 \end{array}$$

EXAMINATION PAPERS.

JANUARY, 1841.

FIRST DIVISION.—(A)

1. IN 365 days, 5 hours, 48 minutes, how many minutes?
Ans. 525948 minutes.

2. A person's salary is £191. 12*s*. 6*d*. per annum; what ought he to receive for 60 days' service?
Ans. £29. 6*s*. $1\frac{71}{73}$*d*.

3. Add together $3\frac{1}{2}$, $4\frac{1}{3}$, $5\frac{1}{4}$, $\frac{3}{4}$ of $\frac{7}{8}$ and $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{5}{8}$.
Ans. $13\frac{27}{32}$.

4. What fraction of a crown is 2*s*. 9*d*., and what decimal of £5. is 3*s*. 6*d*.? Ans. $\frac{11}{20}$, and ·035.

5. Multiply 2·5 by ·0025, and divide 16·9 by ·0013, and by 1·3. Ans. ·00625, 1300, and 13.

6. Find the value of ·3756 of a £1, and reduce $\frac{1}{800}$ to its equivalent simple decimal. Ans. 7*s*. 6·144*d*. and ·00125.

7. If 27 bushels, $2\frac{1}{4}$ pecks, cost £10. 7*s*. $2\frac{1}{4}$*d*., what is the price of a bushel and a half? Ans. 11*s*. 3*d*.

8. Given that the circumference of a circle is to its diameter as 3·1416 to 1; find (in feet and inches) the circumference of a circle, whose diameter is $22\frac{1}{2}$ feet.
Ans. 70 ft. 8·232 in.

9. Find the value of 37 cwt. 2 qrs. 3 lbs. at £3. 7*s*. 10*d*. per cwt. Ans. £127. 5*s*. $6\frac{45}{56}$*d*.

10. Distinguish between interest and discount. Find the difference between the interest and discount of £500., for two years; allowing $4\frac{1}{2}$ per cent. Ans. £3. 14*s*. $3\frac{81}{109}$*d*.

11. Extract the square root of $1\frac{56}{169}$, and the cube root of 1953125. Ans. $1\frac{2}{13}$ and 125.

12. Which is the most advantageous stock to invest in; the 3 per cents. at $89\frac{1}{2}$; or the $3\frac{1}{2}$ per cents. at $98\frac{1}{2}$?

Ans. The $3\frac{1}{2}$ per cents.

13. A room is $29\frac{1}{2}$ feet long, and $14\frac{1}{4}$ feet wide; what will be the expense of carpeting it, at 3*s*. 6*d*. per yard, the carpeting being $\frac{5}{8}$ of a yard wide? Ans. £13. 1*s*. $6\frac{4}{5}$*d*.

14. Find the value of $\dfrac{3x^2-2xy+5y^2+5z^2+2yz+2xz}{4x^2+2xy+3y^2+2z^2+yz-xz}$, when $x=1$, $y=2$, and $z=3$; and subtract $(n-2).(n-3).(n-4)$ from $(n-1).n.(n+1)$. Ans. 2 and $9n^2-27n+24$.

15. Add together

$\dfrac{x-1}{x+1}$, $\dfrac{x+1}{x-1}$, $\dfrac{3x-2}{x+3}$ and $\dfrac{3x-4}{x-2}$. Ans. $\dfrac{8x^4-x^3-24x^2+5x-4}{(x^2-1)(x+3)(x-2)}$.

16. Divide $12x^4-20x^3y+27x^2y^2-18xy^3+4y^4$ by $4x^2-4xy+y^2$. Ans. $3x^2-2xy+4y^2$.

17. Give the geometrical definition of proportion.

18. If quantities are algebraically proportional they are geometrically so.

Obs. For the sake of simplicity the remainders in Examples 2, 9, &c., have been set down as fractions of a *penny*. It will be necessary however always to reduce this to farthings, and then the remainder will be a fraction of a *farthing*.

FIRST DIVISION.—(B)

1. Divide £63. 9*s*. $6\frac{3}{4}$*d*. by 27. Ans. £2. 7*s*. $0\frac{1}{4}$*d*.

2. The value of a mark being 13*s*. 4*d*., and that of a moidore 27*s*.; how many half-crowns are there in 30 marks and 40 moidores together? Ans. 592 half-crowns.

3. What is the tax upon £302. 3*s*. 7*d*. when £429. 8*s*. 3*d*. is rated at 13*s*. 6*d*.? Ans. 9*s*. 6*d*.

4. What is the exact value of

$$\left\{2\tfrac{3}{4}+\frac{5}{2}\text{ of }\frac{7}{3\frac{4}{5}}-\frac{1\frac{2}{3}}{2\frac{1}{2}}\right\}\div 1\tfrac{77}{228}.$$ Ans. 5.

5. Multiply .323756 by .0038, and divide 21 by .0035.

Ans. ·0012302728 and 6000.

6. Reduce 6*s*. $3\frac{1}{4}$*d*. to the decimal of a pound, and find the value of .0324 of a guinea. Ans. $.313541\dot{6}$ and 8·1648*d*.

7. Distinguish between a rule of three direct and a rule of three inverse. If a person travelling $13\frac{3}{4}$ hours a day perform a journey of $27\frac{5}{8}$ days, in what length of time will he perform the same if he travel $10\frac{2}{7}$ hours a day?

Ans. $36\frac{2141}{2304}$ days.

8. Required the amount of £320. 15*s*. for 2 years and 35 days at $4\frac{3}{4}$ per cent. per annum, simple interest.

Ans. £352. 13*s*. $7\frac{1063}{1460}$*d*.

9. What is the discount of £63. 10*s*. due 15 months hence at $4\frac{1}{2}$ per cent? Ans. £3. 7*s*. $7\frac{101}{169}$*d*.

10. Required the vulgar fraction which is equivalent to $.4166\dot{6}$, and its square root to 6 decimal places.

Ans. $\frac{5}{12}$ and ·645497.

Extract the cube root of 42144192. Ans. 348.

11. A cistern of water whose capacity is 20 gallons, is supplied by two spouts, one of which alone would fill it in 2 hours, the other in three hours. It has also a discharging spout which would empty it alone in 4 hours. Supposing the cistern empty and all the spouts open; in what time will it be just filled? Ans. $1\frac{5}{7}$ hours.

12. A person transfers £3000. stock from the 3 per cent. consols at $89\frac{3}{8}$ to the reduced $3\frac{1}{2}$ per cents. at $98\frac{1}{4}$; what is the alteration in his income?

Ans. He gains £5. 10*s*. $3\frac{87}{1311}$*d*. per annum.

13. How many square feet and square inches remain out of 313 square feet of carpeting, after covering a room 16 feet 9 inches long, and 12 feet 11 inches broad: what is the price of the required carpet at 3*s*. 6*d*. a yard?

Ans. 96 sq. ft. 93 sq. in., and £4. 4*s*. $1\frac{47}{72}$*d*.

14. If an expression within brackets whose terms are

connected by the signs + and − be preceded by the sign + or −, how is it affected in each case when the brackets are taken away?

Take away the brackets in $2x - 3y - [(5x + 4y) + 3x + \{y - 9x - (2y - x) + (x - y)\}]$. Ans. $x - 5y$.

15. Divide $x^7 + y^7$ by $x + y$, also $12x^2y^{-1} - 4x^{-3}y^6 + 8y - 6x^{-5}y^4 + 18xy^7 - 36x^4y^2$ by $2x^{-2}y^3 - 4xy^{-2}$.
Ans. $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$, and $9x^3y^4 - 2y^5x^{-1} - 3yx^{-3}$.

16. Shew that

$$a(x+y) + \left\{\frac{1}{a-x} - \frac{1}{a-y} + \frac{x}{(a-x)^2} - \frac{y}{(a-y)^2}\right\} \div \left\{\frac{1}{(a-y)(a-x)^2} - \frac{1}{(a-x)(a-y)^2}\right\} = 2a^2,$$

also that $\dfrac{1}{x + \dfrac{1}{y + \dfrac{1}{z}}} \div \dfrac{1}{x + \dfrac{1}{y}} - \dfrac{1}{y(xyz + x + z)} = 1.$

17. Give the algebraical and geometrical definitions of proportion. Shew that if $a : b :: c : d$, then $ad = bc$ and the converse.

18. Explain what is meant by a quantity varying as another, directly or inversely, or as two others jointly,

If $x - a \propto y - b$, and when $x = m$, $y = n$; find the constant which connects $x - a$ with $y - b$. Ans. Constant $= \dfrac{m-a}{n-b}$.

SECOND DIVISION.—(A)

1. How many grains of gold are there in 9 lbs. 11 oz. 13 dwts. 20 gr.? Ans. 57452 grains.

2. Find how often £24. 11*s*. 6¼*d*. is contained in £8060. 18*s*. 10*d*. Ans. 328 times.

3. If a tradesman with a capital of £2000 gain £50 in 3 months, what sum will he gain with a capital of £3000 in 7 months? Ans. £175.

4. Prove that the sum of the fractions $1\frac{1}{26}$ and $\frac{1}{1\frac{4}{9}}$ is equal to 5 times their difference.

Ans. Their sum equals $\frac{45}{26}$, and their difference equals $\frac{9}{26}$.

5. Add together $\frac{3}{7}$ of £15, $\frac{1}{4}$ of $\frac{1}{2\frac{4}{5}}$ of $\frac{4}{5}$ of £1. 12*s.* and $\frac{4}{7}$ of 3*d.*; reduce the result to the decimal of £10.

Ans. £6. 11*s.*, and ·655.

6. Divide .0068 by 340 and 314 by .0005.

Ans. ·00002 and 628000.

7. Find the value of 5 cwt. 3 qrs. 16 lbs. at £3. 7*s.* 6*d.* per cwt. by practice. Ans. £19. 17*s.* $9\frac{3}{14}$*d.*

8. A watch which is $4^{m}.\ 8\frac{9}{32}''$ too fast at half-past nine A. M. on Tuesday, loses $2^{m}.\ 45''$ daily; what will be the time indicated by the watch at a quarter past five P. M. on the following Friday? Ans. 10 minutes past 5.

9. A person possessing $\frac{3}{14}$ of an estate sold $\frac{2}{7}$ of $\frac{1}{3\frac{1}{5}}$ of his share for £$120\frac{5}{8}$; what would $\frac{1}{5}$ of $\frac{3}{16}$ of the estate sell for at the same rate? Ans. £236. 8*s.* 6*d.*

10. What is the difference between interest and discount? Which of the two is the greater, and on what account? Required the present worth of £35 due 4 months hence at $4\frac{1}{2}$ per cent. Ans. £34. 9*s.* $7\frac{175}{203}$*d.*

11. Required the compound interest of £130 in 3 years at 4 per cent. per annum. Ans. £16. 4*s.* $7\frac{473}{625}$*d.*

12. Extract the square root of .0001695204 and the cube root of 34012224. Ans. ·01302 and 324.

13. A passenger-train which runs at the rate of 36 miles an hour leaves a given station two hours after a luggage-train whose rate is 30 miles, and overtakes it at the railway terminus; required the distance between the station and terminus, and the time which elapses during the passage of each train between them.

Ans. 360 miles, and the times are 10 and 12 hours.

14. The length, breadth, and thickness of a piece of timber are respectively 94 ft. 6 in., 5 ft. and 2 ft. 5 in.; how many solid feet and inches does it contain? What is its price at $8\frac{3}{4}d.$ per solid foot?

Ans. 1141 c. ft. 1512 c. in., and the price is £41. 12*s*. $7\frac{13}{32}d.$

15. Divide $40ab^4 + 49a^3b^2 - 65a^2b^3 - 25a^4b + 6a^5 - 25b^5$ by $5ab^2 + 3a^3 - 8a^2b - 5b^3$,

and $\frac{x^4}{2} - \frac{x^3}{12} - \frac{11x^2}{72} + \frac{61}{144}x - \frac{1}{12}$ by $\frac{3x^2}{2} + \frac{5x}{4} - \frac{1}{3}$.

Ans. $2a^2 - 3ab + 5b^2$ and $\frac{x^2}{3} - \frac{x}{3} + \frac{1}{4}$.

16. Reduce to its simplest form

$$\frac{3}{2x-3} - \frac{2x+15}{4x^2+9} - \frac{2}{2x+3},$$

and shew that $\frac{4}{x^2+1} - \frac{1+2x}{(x^2+1)^2} - \frac{4}{x^2+x+1} + \frac{1}{(x^2+x+1)^2} = \frac{2x^5 - x^2}{(x^2+1)^2 . (x^2+x+1)^2}$.

Ans. $\frac{18(2x+15)}{16x^4-81}$.

17. Give the algebraical definitions of ratio and proportion.

If $a : b :: c : d$ and $b : e :: d : f$, then $a : e :: c : f$.

18. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

SECOND DIVISION.—(B)

1. ADD together $\frac{3}{5}$ of £3. 7*s*. 6*d*. and $\frac{2}{3}$ of $\frac{1}{4}$ of $4\frac{1}{2}$ guineas. Ans. £2. 16*s*. 3*d*.

2. The property in a town is assessed at £60,000; what must be the rate in the pound in order that £2500 may be raised? Ans. 10*d*.

3. Find the simple fraction which is equal to the difference of $\frac{1}{3}$ of $3\frac{7}{8}$ and $\frac{1}{4}$ of $5\frac{1}{4}$. Ans. $\frac{1}{48}$.

4. What fractional part of £1. 6*s*. 3*d*. is 15*s*. 9*d*.?

Ans. $\frac{3}{5}$.

5. Which is the greater ·0231 of a guinea or ·19 of a half-crown?

Ans. The first equals 5·8212*d*, and the second equals 5·7*d*.

6. Divide 58546·73 by ·23141. Ans. 253000.

7. What decimal is equivalent to the sum of ·47, ·013 and $\frac{\cdot 00625}{\cdot 025}$? Ans. ·733.

8. If $3\frac{3}{4}$ shares be worth £27. 10*s*., what are $4\frac{5}{8}$ shares worth? Ans. £33. 18*s*. 4*d*.

9. Find the value of 31 cwt. 3 qrs. 3 lbs. at £3. 5*s*. 10*d*. per cwt. Ans. £104. 11*s*. $11\frac{37}{56}$*d*.

10. What will £500 amount to in 2 years, allowing compound interest at the rate of $4\frac{1}{2}$ per cent. per annum?

Ans. £546. 0*s*. 3*d*.

11. A person having £5000 invests in the Dutch $2\frac{1}{2}$ per cents. at $48\frac{1}{2}$; what is his annual income?

Ans. £257. 14*s*. $7\frac{65}{97}$*d*.

12. Shew that $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ is very nearly $= 10$.

13. A cubic foot of wood weighing 12 lbs., what is the weight of a beam whose length, width and depth are 24, $2\frac{3}{4}$ and $2\frac{1}{2}$ feet? Find also its value at 3*s*. 4*d*. per cubic foot.

Ans. 17 cwt. 2 qrs. 20 lbs., and the value is £27. 10*s*.

14. Prove that

$$\frac{1}{3}\,\frac{1-2x}{x^2-x+1}+\frac{1}{2}\,\frac{1+x}{x^2+1}+\frac{1}{6(x+1)}=\frac{1}{(x^3+1)(x^2+1)}.$$

15. Multiply $3x^3-4x^2y+4xy^2-y^3$ by $2x^2-xy-y^2$.

Ans. $6x^5-11x^4y+9x^3y^2-2x^2y^3-3xy^4+y^5$.

16. If $a : b :: c : d$, then $a+b : b :: c+d : d$,

and $a : c :: b : d$.

17. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

18. Explain what is meant by a quantity varying directly or inversely as another, or jointly as two others.

JANUARY, 1842.

FIRST DIVISION.—(A)

1. FIND the sum of $3\frac{5}{12}$, $7\frac{1}{2}$, $8\frac{2}{3}$, $4\frac{1}{6}$, and $2\frac{1}{4}$. Ans. 26.

2. Reduce $\frac{3}{11}$ of 16*s*. $0\frac{1}{2}$*d*. to the fraction of 17*s*. 6*d*.
Ans. $\frac{1}{4}$.

3. Determine the continued product of

$$\frac{32}{51}, \frac{85}{112}, \frac{189}{207}, \text{ and } \frac{23}{36},$$

expressing the result in its lowest terms. Ans. $\frac{5}{18}$.

4. Add together $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{8}$, and $\frac{1}{5}$ of $\frac{3}{3}$ of $\frac{15}{8}$, and $\frac{1}{4}$.
Ans. $1\frac{5}{16}$.

5. What incomes respectively will £5000 of $3\frac{1}{2}$ per cent. stock, and £5000 invested in $3\frac{1}{2}$ per cent. stock, at $102\frac{2}{3}$ produce? Ans. £175. and £170. 9*s*. $1\frac{1}{11}$*d*. respectively.

6. Divide 7.001722 by .0031, and prove the truth of the result by vulgar fractions. Ans. 2258·62.

7. What sum will purchase 500 vols. at 2*s*. 3*d*. each, 300 at 5*s*., 260 at 7*s*. 6*d*., and 30 at £1. 1*s*.? Ans. £260. 5*s*.

What will this cost be reduced to if a discount of 10 per cent. be allowed? Ans. £234. 4*s*. 6*d*.

8. What is the cost of 7 cwt. 2 qrs. 14 lbs. at £1. 9*s*. 2*d*. per cwt.? Ans. £11. 2*s*. $4\frac{3}{4}$*d*.

9. What is the value at 16 of a legacy of £1000 payable at 21, allowing simple interest at 4 per cent.?
Ans. £833. 6*s*. 8*d*.

Shew how the calculation must be made when compound interest is allowed.

10. What will £204. 5*s*. amount to in 3 years, allowing $4\frac{1}{2}$ per cent. per annum compound interest?
Ans. £233. 1*s*. $7\frac{39}{40}$*d*. nearly.

11. Extract the cube root of 10000000, given log. 2.15435 $= .\dot{3}$. Ans. 215·435.

12. Extract the square root of 49.14290404. Ans. 7·0102.

13. Find the continued product of $a+x$, $b+x$, and $c+x$, and deduce from it the value of $(a+x)^3$.

Ans. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$, and $x^3+3ax^2+3a^2x+a^3$.

14. Explain fully the meaning of the signs $+$, $\div$, $\sqrt{}$, and $\propto$.

15. If $a=1$, $b=\frac{2}{3}$, $x=7$, and $y=8$, find the value of

$$5(a-b)\sqrt[3]{(a+x)y^2}+a-b\sqrt{(a+x)y}.$$ Ans. 9.

16. Multiply $a^2-2ax+bx-b^2$ by $b+ax$.

Ans. $a^2b+(a^3-ab^2-2ab+b^2)x+(ab-2a^2)x^2-b^3$.

17. Give the algebraical definition of proportion, and shew that it includes the geometrical definition. If 3, x and 1083 form a continued proportion; find x. Ans. $x=57$.

18. Add together $\dfrac{a-b}{b}+\dfrac{2a}{a-b}-\dfrac{a^3+a^2b}{a^2b-b^3}$.

Ans. $\dfrac{b}{a-b}$.

19. Find by cross multiplication the content of a floor whose length is 22 ft. $8\frac{3}{4}$ in., and breadth 16 ft. $7\frac{1}{2}$ in.

Ans. 377 sq. ft. $125\frac{5}{8}$ sq. in.

Explain the difference between cross multiplication and duodecimal notation.

20. Prove that if $a:b::c:d$ and $b:e::d:f$, then $a:e::c:f$.

FIRST DIVISION.—(B)

1. ADD together £$3\frac{5}{8}$. $7\frac{4}{5}s.$ and $4\frac{3}{4}d.$, and reduce the sum to a fraction of a pound. Ans. £4. 0s. $8\frac{7}{20}d.$ and $4\frac{167}{4800}$.

2. Reduce $\frac{2}{3}$ of 13s. 6d. to the decimal of a pound.

Ans. ·45.

3. Find the product of 3.1416×2.89 and of $\cdot7854\times(\cdot02)^2$.

Ans. 9·079224 and ·00031416.

4. Add together $\frac{4}{11}$ of $\frac{2}{3}$ of $\frac{1}{2}$, $\frac{7}{8}$ of $\frac{4}{5}$ of $\frac{10}{33}$, and $\frac{2}{5}$ of $1\frac{2}{3}$.
Ans. 1.

5. What income is derived from £14000. 3 per cent. stock, and what from as much 3 per cent. stock as £14000. will purchase at 88 per cent.? Ans. £420. and £477. 5*s*. $5\frac{5}{11}$*d*.

6. Divide ·00711009 by 103, and prove the truth of the result by vulgar fractions. Ans. ·00006903.

7. On the price of 25 vols. bought at 3*s*. a vol. the bidder is allowed 5 per cent., on that of 12 others at 5*s*. 3*d*. a vol. $7\frac{1}{2}$ per cent., $2\frac{1}{2}$ per cent. of the auction duty is also paid by the purchaser. What will the books cost? Ans. £6. 12*s*. $11\frac{7}{10}$*d*.
and the cost without auction duty is £6. 9*s*. $6\frac{3}{10}$*d*.

8. What is the cost of 847 articles at 3*s*. $4\frac{1}{4}$*d*. each?
Ans. £142. 0*s*. $11\frac{3}{4}$*d*.

9. Explain the meaning of the terms *interest*, *discount*, *present worth*. Find the present worth of £1250. due at the end of 3 months, allowing $3\frac{1}{2}$ per cent. per annum.
Ans. £1239. 3*s*. $1\frac{207}{269}$*d*.

10. What will £1760. 10*s*. amount to in $2\frac{1}{2}$ years, allowing 4 per cent. per annum compound interest?
Ans. £1942. 4*s*. 9·58*d*. nearly.

11. Extract the square root of .016, and find the hypotenuse of a triangle whose base and perpendicular are respectively 30 and 40. Ans. ·12648 and 50.

12. Extract the cube root of 8242408. Ans. 202.

13. If $a = 16$, $b = 10$, $x = 5$, and $y = 1$, find the value of

$$(x-b)\,(\sqrt{a}-b)+\sqrt{(a-b)\,(x+y)} \text{ and}$$
$$(a-x)^2-(b-x^2)\pm\sqrt{(a-x)\,(b+y)}.$$

Ans. 36, and 147 or 125.

14. Explain fully the meaning of the terms *vinculum* and *square root*, and the signs $-$ $\times$.

15. Find the continued product of $x+y$, $x+z$, and $a+b$, and deduce the value of $(x+y)^3$. Ans. $ax^2+axy+axz+ayz+bx^2+bxy+bxz+byz$, and $x^3+3x^2y+3xy^2+y^3$.

16. Multiply $2b^2 + 3ab - a^2$ by $7a - 5b$.

Ans. $26a^2b - ab^2 - 10b^3 - 7a^3$.

17. Find by cross multiplication the superficial content of a parallelogram whose length is 75 ft. $7\frac{1}{3}$ in. and breadth 38 ft. $3\frac{1}{4}$ in. Why cannot this method with propriety be called duodecimals?

Ans. 2893 sq. ft. $100\frac{5}{6}$ sq. in.

18. Divide

$$\frac{2x+y}{x+y} - \frac{2y-x}{x-y} - \frac{x^2}{x^2-y^2} \text{ by } \frac{2(x^2+y^2)}{x^2-y^2}. \quad \text{Ans. } \frac{2x^2-2xy-3y^2}{2(x^2+y^2)}.$$

19. Shew that if $\frac{a}{b} = \frac{c}{d}$, a, b, c, d are proportionals according to the geometrical definition. What condition does this definition impose as to the magnitudes being of the same kind?

20. Prove that if

$$a : b :: c : d \text{ then } a+b : b :: c+d : d.$$

SECOND DIVISION.—(A)

1. MULTIPLY 22.0001 by 1000.22, and .201 by $\cdot\dot{3}$.

Ans. 22004·940022, and ·067.

2. Find the value of $\frac{365}{366}$ to 5 places of decimals, and reduce 3.1416 to a fraction in its lowest terms.

Ans. ·99726, and $3\frac{177}{1250}$.

3. If 3 cwt. 17 lbs. cost £4. 15*s*. $7\frac{1}{4}$*d*. what will 16 cwt. 3 qrs. 12 lbs. cost?

Ans. £25. 11*s*. 4*d*.

4. Reduce into simpler forms $\dfrac{17\frac{1}{3}}{73}$ and $\dfrac{7\frac{3}{7}}{40\frac{5}{9}}$, and find the quotient of the latter by the former.

Ans. $\frac{52}{219}$ and $\frac{468}{2555}$, and $\frac{27}{35}$.

5. Reduce 24 days 2 hours 8 min. to the fraction of a month of 30 days.

Ans. $\frac{542}{675}$.

6. Find the difference between $\frac{1}{6}$ of £1. and $\frac{5}{7}$ of two guineas.

Ans. £1. 6*s*. 8*d*.

7. What is the simple interest of £236. 6*s*. 8*d*. for $2\frac{1}{2}$ years, allowing 3 per cent. per annum.?

Ans. £17. 14*s*. 6*d*.

8. If a bankrupt pay 3*s*. 4*d*. in the pound, what will be received on a debt of £3678. 16*s*.? By how much will this be reduced if 1 per cent. on the amount realised be deducted?

Ans. £613. 2*s*. 8*d*., and £6. 2*s*. 7·52*d*.

9. Extract the square root of 66.455104. Ans. 8·152.

10. What will £7. 10*s*. a week amount to between the 7th of March and the 4th of July? Ans. £127. 10*s*.

11. What is the cost of paving a court yard 13 ft. 6 in. by 8 ft. 9 in. at 7*s*. 8*d*. per square yard? Ans. £5. 0*s*. 7½*d*.

12. The dividends on a certain amount of 3 per cent. stock accumulated in 13 years to £3081. How much stock was there, and what will it be worth when this fund is sold at 79⅞ per cent? Ans. £7900.; and £6310. 2*s*. 6*d*.

13. Add together

$$\frac{3a-2b}{2c},\ \frac{2b-3c}{3a} \text{ and } \frac{3ab-2bc}{3ac}. \qquad \text{Ans. } \frac{3a^2-2c^2}{2ac}.$$

14. Investigate the rule for the multiplication of algebraic fractions, and find the product of

$$\frac{x(a-x)}{a^2+2ax+x^2} \text{ and } \frac{a(a+x)}{a^2-2ax+x^2}. \qquad \text{Ans. } \frac{ax}{a^2-x^2}.$$

15. Divide $3x^3+4abx^2-6a^2b^2x-4a^3b^3$ by $2ab+x$.

Ans. $3x^2-2a^2b^2-2abx$.

16. Divide a by $1-x$ to 5 terms, and restore the result with its remainder to the form $\frac{a}{1-x}$.

Ans. $a+ax+ax^2+ax^3+ax^4+\frac{ax^5}{1-x}$.

17. Define ratio and proportion.

18. Expand $(a+x)^4$, and prove the truth of the expansion when $a=1$ and $x=2$.

Ans. $a^4+4a^3x+6a^2x^2+4ax^3+x^4$, and when $a=1$, $x=2$, $3^4=81$.

19. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

20. If $y=mx$ where m is constant, then y varies as x.

SECOND DIVISION.—(B)

1. MULTIPLY .0010102 by 109.01, and $1.000\dot{1}$ by .09081.

Ans. ·110121902, and ·09082007991.

2. Reduce 3.1416 to a fraction in its lowest terms, and $\frac{22}{7}$ to a decimal. Ans. $3\frac{177}{1250}$, and $3{\cdot}\dot{1}4285\dot{7}$.

3. What fraction of 15*s*. $7\frac{1}{4}$*d*. is 2*s*. $2\frac{3}{4}$*d*.? Ans. $\frac{1}{7}$.

4. Find the simple fractions equivalent to

$$\tfrac{1}{2}\cdot\frac{\frac{1}{2}-1}{2}\cdot\frac{\frac{1}{2}-2}{3} \text{ and } \tfrac{1}{2}\cdot\frac{\frac{1}{2}-1}{2}\cdot\frac{\frac{1}{2}-2}{3}\cdot\frac{\frac{1}{2}-3}{4}\cdot\frac{\frac{1}{2}-4}{5}.$$

Ans. $\frac{1}{16}$, and $\frac{5}{256}$.

5. If 858 men in 6 months consume 234 quarters of wheat, how many quarters will be required for the consumption of 979 men for three months and a half? Ans. $155\frac{3}{4}$ quarters.

6. A pipe of wine costs £98., and contains 65 dozen and 4 bottles, what does it cost a bottle? How much would the price be increased by expences amounting to 15 per cent. on the first cost?

Ans. 2*s*. 6*d*. per bottle, and the increase would be $4\frac{1}{2}$*d*.

7. Reduce 16 cwt. 1 qr. 21 lbs. to the decimal of a ton.

Ans. ·821875.

8. If £15. 10*s*. be paid on the 1st of January, and the same sum on every 10th day after, how much will be paid during the year? How many days of payment will fall on the same day of the week as the first? Ans. £573. 10*s*., and 6.

9. Find the amount of discount if £250. 15*s*. is paid 9 months before it is due, and 5 per cent. per annum is allowed.

Ans. £9. 1*s*. $3\frac{15}{83}$*d*.

10. Multiply £7. 13*s*. 4*d*. by 365, and divide £106577. 17*s*. 9*d*. by 17. Ans. £2798. 6*s*. 8*d*.; and £6269. 5*s*. 9*d*.

11. Extract the square root of 210405.69. Ans. 458·7.

12. What clear income is derived from an estate of £800. a year, after paying $\frac{1}{9}$th of this sum for expences, and $4\frac{1}{2}$ per cent. on a mortgage of £2000? Ans. £621. 2*s*. $2\frac{2}{3}$*d*.

13. What is the expence of carpeting a room 15 ft. 6 in. by 12 ft. 9 in., the price of the carpet being 7*s.* 8*d.* per yard, and the width of the carpet 24 in.? Ans. £12. 12*s.* $6\frac{1}{4}$*d.*

14. Prove that $x \div \frac{a}{b} = \frac{bx}{a}$, and divide

$\frac{a^3 + 3a^2x + 3ax^2 + x^3}{x^3 - y^3}$ by $\frac{(a+x)^2}{x^2 + xy + y^2}$. Ans. $\frac{a+x}{x-y}$.

15. Find the value of

$\frac{b - 2a}{x - b} - \frac{a - 2b}{x + b} + \frac{3x(a-b)}{x^2 - b^2}$. Ans. $\frac{b(a+b)}{b^2 - x^2}$.

16. Find the value of $a + b\sqrt{x+y} - (a-b)\sqrt[3]{x-y}$ when $a = 10$, $b = 8$, $x = 12$, and $y = 4$. Ans. 38.

17. If $a : b :: c : d$, then $ad = bc$, and the converse.

18. Shew that $\frac{x^2}{4} + \frac{y^2}{4} + z^2 + \frac{xy}{2} + xz + yz$, and $(x+z)^2$ become identical when x and y each $= a$.

19. Define when one quantity is said to vary as another directly or inversely.

20. Find 5 terms of $x \div (1 + x)$, and restore the result with its remainder to the form $\frac{x}{1+x}$.

Ans. $x - x^2 + x^3 - x^4 + x^5 - \frac{x^6}{1+x}$.

JANUARY, 1843.

FIRST DIVISION.—(A)

1. Subtract 407 from 3924, and explain the operations. Ans. 3517.

2. How many half-crowns, sixpences, and fourpences are there in 25 pounds?

Ans. 200 half-crowns, 1000 sixpences, and 1500 fourpences.

3. What will 1 ton 11 cwt. 2 qrs. 14 lbs. cost at £1. 13*s.* 7*d.* per cwt.? Ans. £53. 2*s.* $0\frac{7}{8}$*d.*

4. If a single article cost 3*s.* 7*d.*, how many dozens may be bought for £86. 10*s.*? Ans. $40\frac{10}{43}$.

5. Explain the rule for dividing one fraction by another. Divide $\frac{2}{3}$ of $\frac{9}{10}$ by $3\frac{3}{14}$. Ans. $\frac{14}{75}$.

6. What part of three guineas is half-a-crown? Ans. $\frac{5}{126}$. How much is $\frac{3}{20}$ of a day? Ans. 3 hrs. 36 mins.

7. If 4 men earn fifteen pounds in 20 days, how many men will earn ten guineas in seven days? Ans. 8 men.

8. Explain the nature and use of decimal fractions, and the rule for pointing off decimals in multiplication.

9. Multiply .017 by 910, and divide .0140994 by .0021.
Ans. 15·47 and 6·714.

10. Reduce a guinea and a half to the decimal of a pound, and find the value of .16875 of a pound.
Ans. 1·575 and 3*s.* $4\frac{1}{2}$*d.*

11. Extract the square root of ·001156 and the cube root of 148877000. Ans. ·034 and 530.

12. Define interest and discount. Point out which is the greater of the two, and why?

13. Find the amount of £375. in two years, at $4\frac{1}{2}$ per cent. compound interest. Ans. £409. 10*s.* $2\frac{1}{4}$*d.*

14. How many cubic feet are contained in a beam 20 ft. 4 in. long, 1 ft. 5 in. broad, and 10 in. thick?
Ans. 24 cubic ft. 8 cubic inches.

15. Multiply $2x^3+4x^2+8x+16$ by $3x-6$, and divide $27x^3+8y^3$ by $3x+2y$. Ans. $6x^4-96$ and $9x^2-6xy+4y^2$.

16. Divide $\dfrac{4(a^2-ab)^2}{b.(a+b)^2}$ by $\dfrac{6ab}{a^2-b^2}$,

and reduce to its simplest form the expression

$\dfrac{2}{x}-\dfrac{3}{2x-1}-\dfrac{2x-3}{4x^2-1}$. Ans. $\dfrac{2a(a-b)^3}{3b^2(a+b)}$ and $\dfrac{2}{x(1-4x^2)}$.

17. Give the algebraical definitions of ratio and proportion; and shew that if

$$a : b :: c : d, \text{ then } a+b : b :: c+d : d.$$

18. If quantities be proportional according to the algebraical definition of proportion, they are also proportional according to the geometrical definition.

FIRST DIVISION.—(B)

1. Multiply 83 by 37, and explain the operations. Ans. 3071.

2. In 3000 fourpences, how many half-crowns and half-guineas? Ans. 400 half-crowns, and $95\frac{5}{21}$ half-guineas.

3. Bought 13 cwt. 1 qr. 8 lbs. of tea at £1. 11*s.* 9*d.* per qr., what does the whole cost? Ans. £84. 11*s.* $9\frac{6}{7}$*d.*

4. A factor bought 56 pieces of stuff for £1569. 17*s.* 4*d.*, at 4*s.* 10*d.* per yard, how many yards in each piece? Ans. 116 yds.

5. Multiply $17\frac{2}{3}$ by $\frac{6}{7}$ of $3\frac{1}{2}$. Ans. 53.

6. What part of £20. is half a guinea? and how much is $\frac{15}{16}$ of a cwt.? Ans. $\frac{21}{800}$, and 3 qrs. 21 lbs.

7. If the wages of 8 persons for 21 weeks be £92., what will be the wages of 14 persons for 33 weeks? Ans. £253.

8. Explain the rule for pointing off the figures in the division of decimal fractions.

9. Multiply 12.73 by .045, and divide 20 by .018. Ans. ·57285 and $1111\cdot\dot{1}$.

10. What decimal of a pound is .45 of a guinea, and how much is .255 of a day? Ans. ·4725, and 6 hrs. 7 mins. 12 secs.

11. Find the square root of 16.4025, and the cube root of 50653. 4·05, and 37.

12. What is the discount upon £500., due 15 months hence, at the rate of 4 per cent.? Ans. £23. 16*s.* $2\frac{2}{7}$*d.*

13. Find the amount of £200., in 3 years, at 3 per cent., compound interest. Ans. £218. 10*s.* 10·896*d.*

14. What is the area of a floor, of which the length is 17 ft. 2 in. 3 pts. and the breadth 15 ft. 11 in. 7 pts.? Ans. 274 sq. ft. $58\frac{1}{16}$ sq. in., or 274. 4′. 10″. 0‴. 9⁗.

15. Multiply $\frac{x^2}{6} - \frac{2}{3}xy + \frac{3}{4}y^2$ by $2\frac{x}{y} - \frac{y}{3x}$, and divide $a^3 - 2ab^2 + b^3$ by $a - b$.

Ans. $\frac{x^3}{3y} - \frac{4x^2}{3} + \frac{13xy}{9} + \frac{2y^2}{9} - \frac{y^3}{4x}$, and $a^2 + ab - b^2$.

16. Reduce $\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$ to its simplest form, and find the continued product of

$$\frac{1-x^2}{1+y}, \ \frac{1-y^2}{x+x^2} \text{ and } 1 + \frac{x}{1-x}.$$

Ans. $\frac{9}{(x-1)(x+2)^2}$, and $\frac{1-y}{x}$.

17. Give the geometrical definition of proportion, and shew that if $a : b :: c : d$, and $c : d :: e : f$, then $a : b :: e : f$.

18. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

SECOND DIVISION.—(A)

1. If a person's annual income be £500., what is his income per day? Ans. £1. 7*s*. $4\frac{56}{73}d$

2. In 350 seven shilling pieces, how many half guineas ar there, and how many pounds?

Ans. $233\frac{1}{3}$ half-guineas, and £122

3. What is the price of 53 cwt. 3 qrs. 7 lbs., at £73. 15*s*. (per cwt.? £3970. 0*s*. 4

4. If the carriage of 5 cwt. 7 lbs. for 84 miles cost £3. 18*s*. 4*d*., what will it cost me to have 21 cwt. 1 qr. 14 carried the same distance? Ans. £16. 10*s*.

5. Give a definition of a fraction, and explain the for multiplying one fraction by another.

6. Multiply together $\frac{3}{4}$, $\frac{5}{7}$, $7\frac{1}{5}$, $\frac{2}{3}$ of $3\frac{8}{9}$. And divid $\frac{2}{3}$, by $\frac{8}{9}$ of $\frac{1}{7}$. Ans. 10

7. What is the value of $\frac{7}{9}$ of a pound sterling? and 11 cwt. 1 qr. 6 lbs. to the fraction of a ton.

Ans. 15*s*. $6\frac{2}{3}d$., an

8. A person can perform a journey in $13\frac{1}{3}$ days, when he travels 12 hrs. 45 min. each day; how long will he be in performing the same, when he travels 10 hrs. 12 min. each day?
Ans. $16\frac{2}{3}$ days.

9. Add together 1.05 and 2.303, and explain the operations.
Ans. 3·353.

10. Find the square root of $2\frac{457}{4096}$, and the cube root of 373248.
Ans. $1\frac{29}{64}$, and 72.

11. What is the amount of £150. for four years, at four per cent. compound interest? Ans. £175. 9*s*. 6·908*d*.

12. What will £2153. 10*s*. bank stock cost at $188\frac{7}{8}$ per hundred; half-a-crown[1] a hundred being paid in addition for commission? Ans. £4070. 2*s*. $3\frac{3}{5}$*d*.

13. What are the values of the decimals .6666...and .36565....? Ans. $\frac{2}{3}$, and $\frac{181}{495}$.

14. A floor is to be carpeted, whose length is 41 ft. 11 in., and its width 22 ft. 3 in. 9 pts. Give the number of square feet and inches contained in the whole, and the price of carpeting at 3*s*. per square yard. Ans. 935 sq. ft. $38\frac{1}{4}$ sq. in.; and £15. 11*s*. $9\frac{1}{16}$*d*.

15. Multiply $x^5 - x^4y + xy^4 - y^5$ by $x + y$, and divide $x^6 - y^6$ by $x - y$.
Ans. $x^6 - x^4y^2 + x^2y^4 - y^6$, and $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$.

16. Shew that $(-b)$ subtracted from a gives the result $a + b$, and $-a$ multiplied into $+b$ gives the result $-ab$.

How are such quantities as $\sqrt{3}$, $2\sqrt{2}$, and $\sqrt{-1}$ designated?

17. Add together $\dfrac{1}{4(1+y)}$, $\dfrac{1}{4(1-y)}$ and $\dfrac{1}{2(1+y^2)}$:

and from $\dfrac{3}{4(x-a)}$ subtract $\dfrac{3}{4(x+a)} + \dfrac{a}{2(x^2+a^2)}$.

Ans. $\dfrac{1}{1-y^4}$ and $\dfrac{a(x^2+2a^2)}{x^4-a^4}$.

18. If $a : b :: c : d$, and $c : d :: e : f$, then $a : b :: e : f$.

Explain also what is meant by a quantity varying as another, directly and inversely.

1. We may therefore suppose the stock to cost £180. per hundred.

SECOND DIVISION.—(B)

1. A CARRIAGE load is found to weigh 1 ton 3 cwt. 1 qr. and it consists of 215 equal packages; what is the weight of each? Ans. 12 lbs. 1 oz. $12\frac{124}{215}$ drs.

2. A piece of money is worth 16*s.* 3*d.*; how many guineas are there in 253 such pieces? Ans. $195\frac{65}{84}$ guineas.

3. Find the rent of 22 acr. 3 roods, 27 perches, at £2. 3*s.* 8*d* per acre. Ans. £50. 0*s.* $9\frac{17}{40}$*d.*

4. If a workman earns £17. 6*s.* in $102\frac{1}{2}$ days, how long will he be in earning 50 guineas? Ans. $311\frac{19}{346}$ days.

5. Distinguish between proper and improper fractions.

Shew that, if the numerator of a fraction can be divided by any number, this operation gives a result equivalent to that from multiplying the denominator by the same number.

6. Reduce to its simplest form the expression

$$\left(\frac{3\frac{1}{3}}{7} + \frac{2}{10\frac{1}{2}} - \frac{5}{18} \text{ of } \frac{4}{7}\right) \times 1\tfrac{3}{4}. \qquad \text{Ans. } \tfrac{8}{9}.$$

7. What fractions of a pound are $\frac{3}{14}$ of a penny, and $\frac{15}{56}$ of a guinea respectively? Ans. $\frac{1}{1120}$, and $\frac{9}{32}$.

8. If a piece of cloth be $94\frac{1}{2}$ yards long and $1\frac{1}{8}$ yards broad, how broad is a piece of the same content whose length is $74\frac{2}{3}$ yards? Ans. $1\frac{217}{512}$ yards.

9. Multiply 9.05 by .0105, and explain the reasons for each part of the operation. Ans. ·095025.

10. Reduce $\frac{119}{360}$ to the form of a decimal, and find the value of .0191919... Ans. ·330$\dot{5}$, and $\frac{19}{990}$.

11. Find the square root of $\frac{15}{21}$ to 3 places of decimals, and the cube root of .000008869743. Ans. ·845, and ·0207.

12. A person lays by £230. at the end of each year, and employs the money at $3\frac{1}{2}$ per cent. compound interest; what will he be worth at the end of three years? Ans. £714. 8*s.* $7\frac{31}{50}$*d.*

13. The 3 per cent. consols are at $94\frac{1}{8}$; what annual income is obtained by the investment of £3500, $\frac{1}{8}$ per cent. upon the stock bought being paid for commission?

Ans. £111. 8*s*. $1\frac{1}{3}\frac{5}{7}\frac{1}{7}$*d*.

14. Determine the volume of a cube, one of whose edges is 2 ft. 7 in. Ans. 17 c. ft. 415 c. in.

15. What is the meaning of $a^{\frac{3}{2}}$? and why is

$$-a \times -b = +ab?$$

16. Multiply $4x^2 - 3xy - y^2$ by $3x - 2y$, and divide

$$\frac{x^2}{2a^2} - 4 + \frac{6a^2}{x^2} \text{ by } \frac{x}{2a} - \frac{3a}{x}.$$

Ans. $12x^3 - 17x^2y + 3xy^2 + 2y^3$, and $\frac{x}{a} - \frac{2a}{x}$.

17. Divide $\frac{ax - x^2}{(a+x)^2}$ by $\frac{x^2}{a^2 - x^2}$, and reduce to its simplest form $\frac{5}{2} \cdot \frac{1}{x+1} - \frac{1}{10} \cdot \frac{1}{x-1} - \frac{24}{5} \cdot \frac{1}{2x+3}$.

Ans. $\frac{(a-x)^2}{ax + x^2}$, and $\frac{2x-3}{(x^2-1)(2x+3)}$.

18. If $a : b :: c : d$, then $a + b : b :: c + d : d$.

JANUARY, 1844.

FIRST DIVISION.—(A)

1. ADD $\frac{1}{3}$ of $\frac{2}{5}$ of 4*s*. $4\frac{1}{2}$*d*. to $\frac{2}{3}$ of $\frac{1}{7}$ of a guinea. Ans. 2*s*. 7*d*.

2. Divide .00683468 by 2.0102, and 34 by .17. Find the value of $\frac{2}{5}$ of .001. Ans. ·0034, 200, and ·0004.

3. Find the value of the decimal £15.275; and reduce 6 cwt. 2 qrs. 7 lbs. to the decimal of a ton.

Ans. £15. 5*s*. 6*d*. and ·328125.

4. What is the cost of 11 cwt. 3 qrs. at £4. 13*s*. 4*d*. per ton? Ans. £2. 14*s*. 10*d*.

5. What is the present value of £918 due four years hence; interest being calculated at the rate of 5 per cent. per annum?

Ans. £765.

6. What is the amount of compound interest on £7625 accumulated for 3 years, at 4 per cent. per annum?

Ans. £952. 1*s*. 9·12*d*.

7. The French franc is divided into 100 centimes and the Frankfort florin into 60 kreutzers. When the pound sterling is worth 25.50 francs at Paris, and 11fl. 54kr. at Frankfort, what is the worth of the Napoleon of 20 francs in florins and kreutzers?

Ans. 9 fl. 20 kreut.

8. Extract the square root of 379.8601 and of ·04.

Ans. 19·49 and ·2.

9. What will be the expense of painting a surface measuring 23 ft. 6 in. by 20 ft. at the rate of 4*s*. 6*d*. per square yard?

Ans. £11. 15*s*.

10. If 18 lbs. 3 oz. cost 6*s*. $0\frac{3}{4}$*d*. what will 2 cwt. 1 qr. 3 lbs. cost?

Ans. £4. 5*s*.

11. What is the meaning of the signs +, −, in Algebra?

12. Subtract $\frac{pq-q^2}{p^2-q^2}$ from $\frac{p^2-pq}{p^2-q^2}$, expressing the result in the simplest form.

Ans. $\frac{p-q}{p+q}$.

13. Add together $4ab-x^2$, $3x^2-2ab$, $2ax+2bx$, and divide their sum by $2(a+x)$.

Ans. $2ab+2x^2+2ax+2bx$, and $b+x$.

14. Multiply together $a-x$, a^2-x^2, and $a+x$.

Ans. $a^4-2a^2x^2+x^4$.

15. If $a : b :: c : d$, then $a+b : b :: c+d : d$.

16. What is the geometrical definition of proportion?

FIRST DIVISION.—(B)

1. ADD $\frac{3}{5}$ of $\frac{7}{4}$ of £1. to $\frac{1}{3}$ of $\frac{5}{2}$ of 7*s*. 6*d*. Ans. £1. 7*s*. 3*d*.

2. Multiply .0005 by .016; and reduce $\frac{1}{25}$ of $\frac{1}{6}$ of $7\frac{1}{2}$ to a decimal. Ans. ·000008, and ·05.

3. Reduce $1\frac{42}{69}$ to a fraction in its lowest terms, and find the value of .0125 of £1. Ans. $1\frac{14}{23}$, and 3*d*.

4. Find the cost of 435 lb. 3 oz. of tea at 5*s*. 4*d*. per pound. Ans. £116. 1*s*.

5. Find the income produced by £12600. stock in the 3 per cents.; and its value sterling, stocks being at 95. Ans. £378. and £11970.

6. If £1. sterling be worth 12 florins, and also worth 25 francs 56 centimes; how many francs and centimes is one florin worth? (100 centimes = 1 franc.) Ans. 2 fr. 13 cent.

7. Extract the square root of 9564.84, and the cube root of 2197. Ans. 97·8, and 13.

8. Find the cost of carpeting a room 12 feet 4 inches by 16 feet 3 inches, at 1*s*. 6*d*. per square foot. Ans. £15. 0*s*. $7\frac{1}{2}$*d*.

9. How much will a person who has £900. a year have to pay under an income tax of 7*d*. in the pound? Ans. £26. 5*s*.

10. If 12 men can reap a field in 4 days, in what time can the same work be performed by 32 men? Ans. $1\frac{1}{2}$ days.

11. Give the algebraical definition of proportions, and shew that it includes the geometrical.

12. Add and reduce

$$\frac{3a-4b}{7} - \frac{2a-b-c}{3} + \frac{15a-4c}{12} - \frac{a+64b}{84}.$$

Ans. $a-b$.

13. Multiply together $x-3$, $x-1$, $x+1$, and $x+3$.
Ans. x^4-10x^2+9.

14. Divide $x^4+10x^3+35x^2+50x+24$ by x^2+5x+4.
Ans. x^2+5x+6.

15. If $a : b :: c : d$, and $c : d :: e : f$, then $a : b :: e : f$.

16. Find the value of $\frac{x}{y} - \frac{7}{2}\sqrt{\frac{1+x}{1+y}}$, when $x = 7, y = 1$.

Ans. 0.

SECOND DIVISION.—(A)

1. ADD $\frac{5}{7}$ of $\frac{1}{4}$ of 6*s*. 5*d*. to $\frac{1}{6}$ of $\frac{2}{5}$ of half a crown.
Ans. 1*s*. $3\frac{3}{4}$*d*.

2. Multiply 1.0076 by 2.0038. Divide .004 by .025 and 124 by .0062. Ans. 2·01902888, ·16 and 20000.

3. Reduce $\frac{494}{221}$ to its lowest terms, and $\frac{33}{132}$ to a decimal form. Ans. $2\frac{4}{17}$ and ·25.

4. If 86 lbs. 7 oz. cost £5. 15*s*. 3*d*. what is the price per ounce? Ans. 1*d*.

5. What is the simple interest and the amount of £2833. 6*s*. 8*d*. at the end of $2\frac{1}{2}$ years, interest being allowed at the rate of 3 per cent. per annum? Ans. £212. 10*s*. and £3045. 16*s*. 8*d*.

6. What income is derived from the sum of £4788. invested in the $3\frac{1}{2}$ per cent. at 105? Ans. £159. 12*s*. per annum.

7. When the pound sterling is worth 45 Pauls 9 Baiocchi (Roman) and $25\frac{1}{2}$ francs (French), what is the value of the Napoleon of 20 francs in Roman money? Ans. 36 Pauls.

N.B. 100 centimes = 1 franc, 10 Baiocchi = 1 Paul.

8. Extract the square root of 8281; and of .001 to four places of decimals. Ans. 91 and ·0316.

9. An area measuring 30 ft. 6 in. by 8 ft. 9 in. is to be paved. What will it cost at the rate of 4*s*. 8*d*. per square foot?
Ans. £62. 5*s*. 5*d*.

10. What will be the cost of 1548 articles at 2*s*. 1*d*. each?
Ans. £161. 5*s*.

11. Explain the nature of the operations indicated by the sign × and ÷. Define the terms "proper" and "improper" fraction.

12. Add together $\dfrac{x(a+x)}{a-x}$, $\dfrac{5ax-x^2}{x-a}$, $\dfrac{2a^2}{a-x}$, and subtract $-(a+x)$ from $a-x$. Ans. $2(a-x)$ and $2a$.

13. Multiply $2p-q$ by $2q+p$, and divide $a^3-2a^2bx-3ab^2x^2$ by $a+bx$. Ans. $3pq-2q^2+2p^2$ and a^2-3abx.

14. Give the algebraical definitions of ratio and proportion.

15. If $a:b::c:d$ and $b:e::d:f$ then $a:e::c:f$.

16. What is meant when one quantity is said to vary *directly* as another?

SECOND DIVISION.—(B)

1. Subtract $\frac{1}{7}$ of 3*s*. $2\frac{1}{2}$*d*. from $\frac{7}{5}$ of $\frac{3}{4}$ of a crown.
Ans. 4*s*. $9\frac{1}{2}$*d*.

2. Multiply 21.32 by 10.0103: divide .0048 by 1.2 and .48 by .00012. Ans. 213·419596, ·004 and 4000.

3. Reduce $\frac{15573}{6889}$ to its lowest terms; and 5*s*. to the decimal of 13*s*. 4*d*. Ans. $2\frac{15}{53}$, and ·375.

4. Find the cost of 7 cwt. 2 qrs. 14 lb. at £7. 10*s*. 10*d*. per cwt. Ans. £57. 10*s*. $1\frac{1}{4}$*d*.

5. Find the discount of £127. 2*s*. due in 6 months at 5 per cent. Ans. £3. 2*s*.

6. If £1. sterling be worth 25 francs 60 centimes; and also worth 6 thalers 20 silber groschen; how many francs and centimes is a thaler worth? (one thaler = 30 silber groschen, 1 franc = 100 centimes.) Ans. 3 francs 84 centimes.

7. Extract the square roots of .018225 and of $1\frac{25}{144}$.
Ans. ·135 and $1\frac{1}{12}$.

8. Find the cost of papering a wall 14 feet 3 inches long, and 11 feet 11 inches high; at 1*s*. 4*d*. the square foot.
Ans. £11. 6*s*. 5*d*.

9. The rents of a parish amount to £2360. 10*s*.; and a rate is granted of £160. 16*s*. 8*d*. What portion must be paid by a person whose rental is £590. 2*s*. 6*d*.? Ans. £40. 4*s*. 2*d*. or $\frac{1}{4}$th.

10. Bought goods at $6\frac{1}{2}d.$ per lb. and sold them at £4. 10*s.* per cwt. What is the gain or loss per cwt.?

Ans. The gain is £1. 9*s.* 4*d.* per cwt.

11. When is one quantity said to vary directly or inversely as another?

12. Shew that $\dfrac{1-x+x^2}{1+x-x^2}+\dfrac{6x^3-4x^4}{1+x-x^2}-4x^2+2x=1.$

13. Multiply $x+2y-3z$ by $x-2y+3z$.

Ans. $x^2-4y^2+12yz-9z^2$.

14. Divide a^5+b^5 by $a+b$. Ans. $a^4-a^3b+a^2b^2-ab^3+b^4$.

15. If $a:b::c:d$ then $ad=bc$; and the converse.

16. Find the value of $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}-\dfrac{x-9}{x+2}$ when $x=1$.

Ans. 0.

J. HALL, PRINTER, CAMBRIDGE.